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The Effects of Migration and Fear on the Stability of an Eco-Epidemiological Model with Hunting Cooperation among Predators

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ABSTRACT:

Background: In the current study, we construct a 2D eco-epidemiological model in which prey populations on one hand, fear and competition among prey populations internalized, and evolution of spatial structure merit to predator (P) zone vs. protecting-shield (PRQ) in between. Individuals of the predator population can be either healthy or susceptible-infected (SI) individuals, and the disease spreads by contactbetween SIand Sindividuals. Hunting cooperation between predators and that disease propagates in the predator's population.

Methods: The proposed model incorporates the Sokol-Howell functional response to describe predator–prey interactions. The mathematical properties of the model were investigated by establishing its boundedness and uniqueness. All equilibrium points were determined, and the conditions required for their local and global stability were analyzed using appropriate mathematical techniques, including Lyapunov functions. Numerical simulations were also performed using a selected parameter set and varying initial conditions.

Results: The analysis identified all equilibrium points of the model and established the necessary conditions for their local stability. Under the corresponding conditions, all equilibrium points were shown to be locally stable. Furthermore, the global stability analysis based on suitable Lyapunov functions confirmed the stability properties of the system. Numerical simulations supported the analytical findings and demonstrated the effects of SI disease on the predator–prey dynamics .

Conclusion: The findings demonstrate that incorporating fear, internal rivalry, migration, and SI disease transmission provides a more comprehensive representation of predator–prey dynamics in protected and predator zones. The analytical and numerical results highlight the significant role of disease transmission in influencing population dynamics and support the validity of the proposed eco-epidemiological model.

INTRODUCTION

The growing popularity of Biomathematics is made possible by its wide range of applications in ecology and environmental sciences, epidemiology, population dynamics. Lots of researchers have focused their work around the predator-prey model, which is one of the most essential population models. This is frequently employed to study species interactions (Aguiar *et al.*, 2022; Zhuo et al., 2025).Mathematical modeling is becoming increasingly significant across the sciences. While most people are aware of the significant influence mathematics, throughout the epidemic of the past 18 months, society at large has discovered that mathematical modeling has broader applications from domains as diverse as physics, biology, medicine, ecology, epidemiology, and even sociology(Marion et al., 2008 ; Tasiu et al., 2024).

Mathematical modeling is a powerful method to understand complex phenomena and systems through mathematical descriptions. It enables us to study the dynamics of complex systems and predict their responses under specific circumstances — equilibria that are stable or unstable, bifurcation, persistence. An immense number of mathematical models have been suggested and investigated(Chattopadhyay and Arino , 1999; Annavarapu, R.B. and M. Sharma, 2022; Majeed , 2024).The predator-prey model is a fundamental tool in scientific biology, particularly for determining how populations interact in their natural environment. Population biology is a subject of natural science that analyzes the forces shaping populations. The scientific theory of animal migration proposes that living creatures seek better living conditions (Hameed and H. M. Al-Saedi,2021;Alabacy et al., 2024; Majeed, 2024).

Moreover, Radiea and Majeed (2021) studied migration happens in a prey-predator system with several patches. Various reasons, including as competition, age, gender, food scarcity, climate, and season, may induce animals to migrate from one patch to another. Many academics (Samuel , 2020; Majeed and Ghafel,2022; Zhong et al., 2025) have researched the significance of fear in biology and ecology (Aziz and Majeed, 2024; Majeed, 2024). Fear has a greater impact on decreasing growth, according to the development of mathematical models (Hadi and Majeed,2024). Since then, numerous researchers have considered the fear effect of predators in dynamical studies of prey-predator relationships.

However, Hassan and Majeed (2025) studied an ecological model with Sokol-Howell functional response, This is a variation of Holling type III. According to the Sokol-Howell model, predator consumption ratios rise with prey density before plateauing due to constraints such as the time required to locate and consume each prey item. Predators cannot constantly increase their consumption ratios, even when prey is abundant (Saha and Samanta, 2021). We will use the Sokol-Howell functional response to investigate predator-prey behavior and estimate the effects of both infected and uninfected predators.Enatsu and et al. The research in (Enatsu et al., 2024) investigates a predator-prey model with cooperative specialist predators. Likewise, pairs of expert predators have been found to capture a higher percentage of their final target prey through collaboration. It is this that gives rise to the prey-predator models with co-operative hunters (Shawka and Majeed, 2024). A study on hunting cooperation and fear among predators of prey Belew=021C0 and Melese (2022) an eco-epidemiological model was proposed and investigated by Shawka, and Majeed (2025) with internal competition captured through fear.

A survey of things like the spread, pathogenesis, treatment and other influences which concern us directly. Eco-epidemic models study disease dynamics in ecosystems of competing populations(Shalan , 2020; Shaikh et al , 2020; Moustafa, 2020).Medicine's primary mathematical models are first-order ordinary differential equations , which have been researched using both qualitative and quantitative methods(Anderson and May, 1986; Akhtar et al., 2024).

Singh and Kaladhar (2024) investigate mathematical models of epidemics, which are essentially population models, although epidemiology is about sickness outcomes (and possibly control) (Venturino, 2016) . Finding such population models is differentiated than the one where some states are pathogenic agents (Benhamou, 2024) thus it makes sense to explore this.Scientisits used to study a myriad of predator-prey models, essentially determining if the disease is transmitted by prey, the predator or both (Matlob and S. Naji,2025). The disease partition predator model into two categories: Infected predators and

Healthy predators also considers the infected predator's organisms. This has put many species at risk of extinction due to epidemics (Ghadeer and M. A. Mohammed,2021).Furthermore, Bezabih and et al. (2021) investigated how disease spreads from diseased predators to healthy individuals by direct contact, contributing to a better understanding of the epidemiological mechanisms that drive ecosystem dynamics (Kebedow et al., 2025).

Noutchie and et al. (2024) provide a novel predator-prey model that adds Susceptible-Infected (SI) disease dynamics within the predator population; infected predators are more likely to die as a result of disease-induced death (Cojocaru et al., 2020; Joudha and S. N. A. Al-Azzawi, 2025).

This research created and investigated an eco-epidemiological model that incorporates prey in the protective zone, prey in the predatory zone, susceptible and infected predators, and Sokol-Howell functional responses to characterize feeding processes. Our research explores the impact of migration, fear, hunting, and sickness on the dynamics of solutions.

MATHEMATICAL MODEL

This paper proposed a model of (prey in a protected area, prey in an unprotected area, healthy predator and infected predator), the total population densities at time (T) are referred to $E_{1r}(T), E_{1p}(T), E_{2s}(T), E_{2I}(T)$, respectively with Sokol-Howell functional response for analysis.

$$\frac{dE_{1r}}{dT} = g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - \rho_1 E_{1r} + \rho_2 E_{1p} - m_1 E_{1r} \ ,$$

$$\frac{dE_{1p}}{dT} = \frac{g_2 E_{1p}}{1 + f E_{2s}} - \mathrm{c}\, E_{1p}^2 + \rho_1 E_{1r} - \rho_2 E_{1p} - \frac{a E_{1p} E_{2s}}{\alpha + \gamma E_{1p}^2} - \delta_1 E_{2s}^2 E_{1p} - m_2 E_{1p} \ , \qquad (1)$$

$$\frac{dE_{2s}}{dT} = \frac{\mathrm{b} E_{1p} E_{2s}}{\alpha + \gamma E_{1p}^2} + \delta_2 E_{1p} E_{2s}^2 - F E_{2s} E_{2I} - m_3 E_{2s} \ ,$$

$$\frac{dE_{2I}}{dT} = \mathrm{F} E_{2s} E_{2I} - \mathrm{d} E_{2I} \ .$$

With the following initial condition $E_{1r}(0) \geq 0$,$E_{1p}(0) \geq 0$,$E_{2s}(0) \geq 0$ and $E_{2I}(0) \geq 0$. The biological meaning of the characteristics in system (1), Table 1:

Table 1 Biological description,of system's (1) parameters

Parameters,	**Bio-description,**
$g_i > 0, i = 1,2$	The intrinsic growth ratios of prey in the protection zone and predatory zone, respectively
$k > 0.$	Carrying capacity of the prey in protection zone
$\rho_i > 0, i = 1,2$	The prey unit migration and is the prey unit emigration of the prey population in protection zone and predatory zone, respectively,
$m_i > 0, i = 1,2,3$	Natural death ratios for prey in the protection ,zone, predatory zone and the susceptible predator, respectively
$f > 0$.	The fear ratio in predatory zone of the susceptible predator,
$c > 0.$	Internal competition ratio of prey, in predatory zone
$a > 0.$	Attack ratio of prey in predatory zone by the susceptible predator
$b > 0.$	Food transformation ratio for prey in predatory zone
$\alpha > 0.$,	Half-saturation constant,
$\gamma > 0.$	Inverse measure of inhibitory action,

$\delta_i > 0, i = 1,2$	Hunting cooperation ratios of prey,in predatory zone and susceptible predator, respectively,
$F > 0.$	The disease transmission between the susceptible and infected predators, respectively
$d > 0.$	Natural the mortality ratio of the afflicted predator, due to disease

It is, anyway easy to verify that all the functions with partial derivatives in system (1) are continuous,in R_+^4; we mean in E_1r,,E_p1,〗 E_2s and E_2I variables. Hence they are Lipschitzian functions, that meaning one unique solution of system (1) as function of initial conditions. The following theorem shows that the system is uniformly bounded.

Theorem 2.1: All solutions of system (1) that start in R_+^4 are uniformly bounded.

Proof: Let $\left(E_{1r}(t)\,,E_{1p}(t)\,,E_{2s}(t)\,,E_{2I}(t)\right)$ be any solution of system (1) with non-negative initial conditions $\left(E_{1r}(0)\,,E_{1p}(0)\,,E_{2s}(0)\,,E_{2I}(0)\right) \in R_+^4$.

Assume that $E' = E_{1r}(t) + E_{1p}(t) + E_{2s}(t) + E_{2I}(t)$, then by drive E' along the solution of system (1), imply

$$\frac{dE'}{dt} = g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - \rho_1 E_{1r} + \rho_1 E_{1r} - m_1 E_{1r} + \frac{g_2 E_{1p}}{1 + f E_{2s}} - \mathrm{c}\, E_{1p}^2 -$$

$$\rho_2 E_{1p} + \rho_2 E_{1p} - \frac{a E_{1p} E_{2s}}{\alpha + \gamma E_{1p}^2} + \delta_2 E_{1p} E_{2s}^2 - \delta_1 E_{2s}^2 E_{1p} + \frac{\mathrm{b} E_{1p} E_{2s}}{\alpha + \gamma E_{1p}^2} - m_2 E_{1p}$$

$$+\mathrm{F} E_{2s} E_{2I} - \mathrm{d} E_{2I} - F E_{2s} E_{2I} - m_3 E_{2s}.$$

Now, by the biological facts, always $b < a$ and $\delta_2 < \delta_1$ hence it is obtained that

$$\frac{dE'}{dt} \le g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - m_1 E_{1r} + g_2 E_{1p}\left(1 - \frac{c}{g_2} E_{1p}\right) - m_2 E_{1p} - m_3 E_{2s} - \mathrm{d} E_{2I}.$$

Thus,

$$\frac{dE'}{dt} \le g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) + g_2 E_{1p}\left(1 - \frac{c}{g_2} E_{1p}\right) - m' E',$$

where $m' = \min\,\{m_1\,, m_2\,, m_3\,, \mathrm{d}\}$.

Now, since the logistic function $f_1(E_{1r}) = g_1 E_{1r} - \frac{g_1 E_{1r}^2}{k}$ is bounded above by the constant $\frac{k g_1}{4}$.

Also, since the logistic function $f_2(E_{1p}) = g_2 E_{1p}\left(1 - \frac{c}{g_2} E_{1p}\right)$ is bounded above by the constant $\frac{g_2^2}{4c}$, so it is observed that

$$\frac{dE'}{dt} + m'E' \le R\ , \quad \text{where } \mathrm{R} = \frac{1}{4}\left(\mathrm{k} g_1 + \frac{g_2^2}{c}\right).$$

Then by solving the above differential inequality for the initial value $E(0) = E_0$, it is obtained that

$$E(t) \le \frac{R}{m'} + \left(E_0 - \frac{R}{m'}\right) e^{-m't}.$$

Then,

$$\lim_{t\to\infty} E(t) \le \frac{R}{m'}.$$

So, $0 \le E(t) \le \frac{R}{m'}$,the solutions are evenly bounded.

EXISTENCE OF EQUILIBRIUM POINTS (EPs),

Its improved that the system (1) has discussed all the points of equilibrium that can check the conditions of existence which are summarized below:

- The (EP) $E_0 = (0,0,0,0)$ always exists.
- The (EP) $E_1 = (\bar{E}_{1r}, 0, 0,0)$, where $\bar{E}_{1r} = \frac{k(g_1-(\rho_1+m_1))}{g_1}$ exists if the following condition holds

$$g_1 > \rho_1 + m_1. \tag{2}$$

- The (EP) $E_2 = \left(0, \bar{\bar{E}}_{1p}, 0,0\right)$ where $\bar{\bar{E}}_{1p} = \frac{g_2-(\rho_2+m_2)}{c}$ exists if the following condition holds

$$g_2 > \rho_2 + m_2. \tag{3}$$

- The (EP) $E_3 = \left(E_{1r}^*, E_{1p}^*, 0,0\right)$ exists uniquely in R_+^2 if the following system:

$$g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - \rho_1 E_{1r} + \rho_2 E_{1p} - m_1 E_{1r} = 0, \tag{4}$$

$$g_2 E_{1p} - cE_{1p}^2 + \rho_1 E_{1r-}\rho_2 E_{1p} - m_2 E_{1p} = 0, \tag{5}$$

has a positive solution

From Equation (4) we have

$$E_{1p} = \frac{E_{1r}[g_1(E_{1r} - k) + k(m_1 + \rho_1)]}{k\rho_2}. \tag{6}$$

by substituting Equation (6) in Equation (5), we find the following equation.

$$\xi_1 E_{1r}^3 + \xi_2 E_{1r}^2 + \xi_3 E_{1r} + \xi_4 = 0, \tag{7}$$

where:

$$\xi_1 = -cg_1^2,$$
$$\xi_2 = 2\text{ck}g_1(g_1 + k(m_1 + \rho_1)),$$
$$\xi_3 = k[ck^2[(m_1+\rho_1)[g_1(2 - g_1) - (m_1 + \rho_1)]] + g_1\rho_2(g_2 + m_2 - \rho_2)],$$
$$\xi_4 = \text{k}[3(m_1 + \rho_1) + k\rho_2[(g_2 + \rho_1\rho_2) + g_1(\rho_2 + m_2)]].$$

Clearly, by Descartes's rule, Equation (7) improved a unique positive root E_{1r} if the following conditions hold:

$$E_{1r} < k, \tag{8}$$
$$k(m_1 + \rho_1) > g_1(k - E_{1r}), \tag{9}$$
$$2 < g_1 < 2(m_1 + \rho_1), \tag{10}$$
$$g_1(2 - g_1) > (m_1 + \rho_1). \tag{11}$$

So, the (EP) $E_3 = (E_{1r}^*, E_{1p}^*, 0,0)$ where $E_{1p}^* = E_{1p}(E_{1r}^*)$ exists with if the above conditions hold.

- The (EP) $E_4 = \left(0, \tilde{E}_{1p}, \tilde{E}_{2s}, \tilde{E}_{2I}\right)$ exists uniquely in R_+^3 if the following system:

$$\frac{g_2}{1 + fE_{2s}} - cE_{1p} - \rho_2 - \frac{aE_{2s}}{\alpha + \gamma E_{1p}^2} - \delta_1 E_{2s}^2 - m_2 = 0, \tag{12}$$

$$\frac{\text{b}E_{1p}}{\alpha + \gamma E_{1p}^2} + \delta_2 E_{1p}\text{E}_{2s} - FE_{2I} - m_3 = 0, \tag{13}$$

$$\text{F}E_{2s} - \text{d} = 0, \tag{14}$$

has a positive solution.

From Equation (14), we have

$$\tilde{E}_{2s} = \frac{d}{F}. \tag{15}$$

by substituting Equation (15) in Equation (12), we find the following equation.

$$ж_1 E_{1p}^3 + ж_2 E_{1p}^2 + ж_3 E_{1p} + ж_4 = 0, \qquad (16)$$

where:

$$ж_1 = -c\gamma F^3(F + fd),$$
$$ж_2 = \gamma F^2[g_2 - (F + fd)(\delta_1 d^2 + m_2 F)],$$
$$ж_3 = -c\alpha F^3(F + fd),$$
$$ж_4 = F^2[F + fd + \alpha g_2] - [ad + \alpha(\delta_1 d^2 + m_2 F)].$$

Clearly, by Descartes's rule, Equation (16) improved a unique positive root say $\tilde{E}_{1p}$ if the following conditions hold:

$$g_2 < (F + fd)(\delta_1 d^2 + m_2 F), \qquad (17)$$

$$F^2[F + fd + \alpha g_2] > [ad + \alpha(\delta_1 d^2 + m_2 F)]. \qquad (18)$$

Now, by substituting $\tilde{E}_{1p}$ in Equation (13), we get

$$\tilde{E}_{2I} = \left(\frac{Fb\tilde{E}_{1p} - (\alpha+\gamma\tilde{E}_{1p}^2)(\delta_2 d\tilde{E}_{1p} + m_3 F)}{F^2\left(\alpha+\gamma\tilde{E}_{1p}^2\right)}\right), \qquad (19)$$

So, $E_4 = \left(0, \tilde{E}_{1p}, \tilde{E}_{2s}, \tilde{E}_{2I}\right)$ exist if the next condition holds:

$$Fb\tilde{E}_{1p} > \left(\alpha + \gamma\tilde{E}_{1p}^2\right)\left(\delta_2 d\tilde{E}_{1p} + m_3 F\right). \qquad (20)$$

- The (EP) $E_5 = \left(\breve{E}_{1r}, \breve{E}_{1p}, \breve{E}_{2s}, 0\right)$ exists uniquely in R_+^3 if the following system: $g_1 E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - \rho_1 E_{1r} + \rho_2 E_{1p} - m_1 E_{1r} = 0, \qquad (21)$
- $\frac{g_2 E_{1p}}{1+fE_{2s}} - cE_{1p}^2 + \rho_1 E_{1r} - \rho_2 E_{1p} - \frac{\alpha E_{2s} E_{1p}}{\alpha+\gamma E_{1p}^2} - \delta_1 E_{1p} E_{2s}^2 - m_2 E_{1p} = 0, \qquad (22)$

$$\frac{bE_{1p}}{\alpha + \gamma E_{1p}^2} + \delta_2 E_{1p} E_{2s} - FE_{2I} - m_3 = 0, \qquad (23)$$

has a positive solution.

From Equation (21), have

$$E_{1p} = \frac{E_{1r}[g_1(E_{1r} - k) + k(m_1 + \rho_1)]}{k\rho_2}. \qquad (24)$$

From Equation (23), have

$$E_{2s} = \frac{bE_{1p} - m_3\left(\alpha + \gamma E_{1p}^2\right)}{\delta_2 E_{1p}\left(\alpha + \gamma E_{1p}^2\right)}. \qquad (25)$$

Now, by subtitling Equation (24) and (25), find the following equation.

$$в_1 E_{1r}^4 + в_2 E_{1r}^3 + в_3 E_{1r}^2 + в_4 E_{1p} + в_5 = 0, \qquad (26)$$

where:

$$в_1 = -\gamma m_3 g_1^2,$$
$$в_2 = [2kg_1(m_3\gamma g_1 + m_1 + \rho_1)],$$
$$в_3 = kg_1[\rho_2 b - k[\gamma g_1 m_3 - (m_1 + \rho_1)(2 + m_1 + \rho_1)]],$$
$$в_4 = b\rho_2 k^2(m_1 + \rho_1 - g_1),$$
$$в_5 = -\alpha m_3 k^2 \rho_2^2.$$

Descartes' rule clearly states that either Eq. (26) has no root, or there are two roots, say, $\breve{E}_{1r}$, $\breve{E}'_{1r}$.

So, $E_5 = \left(\breve{E}_{1r}, \breve{E}_{1p}, \breve{E}_{2s}, 0\right)$ *and* $E_6 = \left(\breve{E}'_{1r}, \breve{E}'_{1p}, \breve{E}'_{2s}, 0\right)$, where $\breve{E}_{1p} = E_{1p}\left(\breve{E}_{1r}\right)$, $\breve{E}'_{1p} = E_{1p}\left(\breve{E}'_{1r}\right)$, $\breve{E}_{2s} = E_{2s}\left(\breve{E}_{1p}\right)$ *and* $\breve{E}'_{2s} = E_{2s}\left(\breve{E}'_{1p}\right)$ exist, where if in addition to condition,(2) the following conditions hold,

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$\breve{E}_{1r} < k,$ (27)
$k(m_1+\rho_1) > g_1(k-E_{1r}),$ (28)
$b\breve{E}_{1p} > m_3(\alpha+\gamma\breve{E}_{1p}^2).$ (29)

- The positive equilibrium point (PEP) $E_7 = (\dot{E}_{1r}, \dot{E}_{1p}, \dot{E}_{2s}, \dot{E}_{2I})$ exists uniquely in R_+^4 if the following system:

$$g_1E_{1r}\left(1-\frac{E_{1r}}{k}\right)-\rho_1E_{1r}+\rho_2E_{1p}-m_1E_{1r}=0, \quad (30)$$

$$\frac{g_2E_{1p}}{1+fE_{2s}}-cE_{1p}^2+\rho_1E_{1r}-\rho_2E_{1p}-\frac{aE_{2s}E_{1p}}{\alpha+\gamma E_{1p}^2}-\delta_1E_{1p}E_{2s}^2-m_2E_{1p}=0, \quad (31)$$

$$\frac{\mathrm{b}E_{1p}}{\alpha+\gamma E_{1p}^2}+\delta_2E_{1p}\mathrm{E}_{2s}-FE_{2I}-m_3=0, \quad (32)$$

$$\mathrm{F}E_{2s}-\mathrm{d}=0, \quad (33)$$

has a positive solution.

From Equation (30), we have

$$E_{1p}=\frac{E_{1r}[g_1(E_{1r}-k)+k(m_1+\rho_1)]}{k\rho_2}. \quad (34)$$

From Equation (33), we have

$$\dot{E}_{2s}=\frac{d}{F}. \quad (35)$$

by substituting Equation (34), (35), and (31), obtain the equation

$$z_1E_{1r}^8+z_2E_{1r}^7+z_3E_{1r}^6+z_4E_{1r}^5+z_5E_{1r}^4+z_6E_{1r}^3+z_7E_{1r}^2+z_8E_{1r}+z_9=0, \quad (36)$$

where:

$$z_1=-\left(1+\frac{fd}{F}\right)\frac{c\gamma g_1^4}{k^4\rho_2^4},$$

$$z_2=-\left(1+\frac{fd}{F}\right)\frac{4c\gamma kg_1^4}{k^4\rho_2^4},$$

$$z_3=\left(\frac{\gamma g_1^3}{k^3\rho_2^3}\right)\left[g_2-\left(1+\frac{fd}{F}\right)\left(\frac{6ck^2g_1}{k\rho_2}-\frac{\delta_1d^2}{F^2}-m_2\right)-1\right]-c\left(1+\frac{fd}{F}\right)\left(\frac{6\gamma k^2g_1^2(m_1+\rho_1)^2}{k^4\rho_2^4}\right),$$

$$z_4=\frac{3\gamma g_1^2(m_1+\rho_1-g_1)}{k^2\rho_2^3}\left[g_2-\left(1+\frac{fd}{F}\right)\left[\frac{\delta_1d^2}{F^2}+m_2+\rho_2\right]\right]$$
$$+\gamma\left(1+\frac{fd}{F}\right)\left[4cg_1\left(\frac{1+(m_1+\rho_1)^2(3g_1-(m_1+\rho_1))}{k\rho_2^4}\right)+\left(\frac{\rho_1g_1^2}{k^2\rho_2^2}\right)\right],$$

$$z_5=\frac{3\gamma g_1^2(g_1-3(m_1+\rho_1))[2-(m_1+\rho_1)]}{k\rho_2^3}\left[g_2+\left(1+\frac{fd}{F}\right)\left(\rho_2-m_2-\frac{\delta_1d^2}{F^2}\right)\right]-\left(1+\frac{fd}{F}\right)\left(\frac{g_1^2}{k\rho_2^2}\right)\left[\frac{c\gamma}{k}+2\gamma\rho_1\right]$$
$$-\gamma\left(\frac{kg_1^4+g_1(m_1+\rho_1)[6-4(m_1+\rho_1)+(m_1+\rho_1)^2]}{k\rho_2^4}\right),$$

$$z_6=\gamma\left(\frac{g_1-[g_1^2-3(m_1+\rho_1)(g_1-(m_1+\rho_1))]+(m_1+\rho_1)^3}{\rho_2^3}\right)\left[g_2-\left(1+\frac{fd}{F}\right)\left(\frac{\delta_1d^2}{F^2}+\rho_2-m_2\right)\right]$$
$$+\left(1+\frac{fd}{F}\right)\left[c\alpha\left(\frac{2g_1[g_1-(m_1+\rho_1)]}{k\rho_2^2}\right)-\gamma g_1\left(\frac{4g_1(m_1+\rho_1)}{k^3\rho_2^4}\right)+\gamma\rho_1\left(\frac{g_1[k+2(m_1+\rho_1)]}{k\rho_2^2}\right)\right],$$

$$z_7=\left(1+\frac{fd}{F}\right)\left[-c\alpha\left(\frac{kg_1[g_1-2(m_1+\rho_1)]+(m_1+\rho_1)}{k\rho_2^2}\right)+\gamma\rho_1\left(\frac{(m_1+\rho_1)[-2g_1+m_1+\rho_1]}{\rho_2^2}\right)\right.$$
$$\left.-(\frac{g_1}{k\rho_2})[\alpha\left(\frac{1}{k}+m_2+\frac{\delta_1d^2}{F^2}\right)]+ad\right]+\frac{\alpha g_1g_2}{k\rho_2}+\frac{3\gamma(m_1+\rho_1)}{k^2\rho_2^4},$$
$$z_8=\frac{m_1+\rho_1-g_1}{\rho_2}\left(\alpha\left[1-\left(1+\frac{fd}{F}\right)\left(\rho_2+m_2+\frac{\delta_1d^2}{F^2}\right)\right]-ad\right)+\left(1+\frac{fd}{F}\right)\left[\alpha\rho_1-\frac{12\gamma g_1^3(m_1+\rho_1)}{k\rho_2^4}\right],$$
$$z_9=c\left(1+\frac{fd}{F}\right)[4\gamma g_1^3k^4(m_1+\rho_1)].$$

Clearly, according to the Descartes's rule, Eq.(36) has a unique positive root say $\dot{E}_{1r}$,by the condition,(2) with the next conditions:

$$g_2<\left(1+\frac{fd}{F}\right)\left(\frac{6ck^2g_1}{k\rho_2}-\frac{\delta_1d^2}{F^2}-m_2\right)+1,\tag{37}$$
$$g_1>\max\left\{\frac{m_1+\rho_1}{3},3(m_1+\rho_1)\right\},\tag{38}$$
$$g_2>\left(1+\frac{fd}{F}\right)\left[\frac{\delta_1d^2}{F^2}+m_2+\rho_2\right],\tag{39}$$
$$4cg_1\left(\frac{1+(m_1+\rho_1)^2(3g_1-(m_1+\rho_1))}{k\rho_2^4}\right)>\left(\frac{\rho_1g_1^2}{k^2\rho_2^2}\right),\tag{40}$$
$$(m_1+\rho_1)<2,\tag{41}$$
$$\rho_2>m_2+\frac{\delta_1d^2}{F^2},\tag{42}$$
$$(m_1+\rho_1)[4+m_1+\rho_1]<6,\tag{43}$$
$$(m_1+\rho_1)^3>(g_1^2-3(m_1+\rho_1)(g_1-(m_1+\rho_1)),\tag{44}$$
$$g_2<\left(1+\frac{fd}{F}\right)\left(\frac{\delta_1d^2}{F^2}+\rho_2-m_2\right),\tag{45}$$
$$\gamma g_1\left(\frac{4g_1(m_1+\rho_1)}{k^3\rho_2^4}\right)>c\alpha\left(\frac{2g_1[g_1-(m_1+\rho_1)]}{k\rho_2^2}\right)+\gamma\rho_1\left(\frac{g_1[k+2(m_1+\rho_1)]}{k\rho_2^2}\right),\tag{46}$$
$$\left(1+\frac{fd}{F}\right)\left[-c\alpha\left(\frac{kg_1[g_1-2(m_1+\rho_1)]+(m_1+\rho_1)}{k\rho_2^2}\right)+\gamma\rho_1\left(\frac{(m_1+\rho_1)[-2g_1+m_1+\rho_1]}{\rho_2^2}\right)\right.$$
$$\left.-(\frac{g_1}{k\rho_2})[\alpha\left(\frac{1}{k}+m_2+\frac{\delta_1d^2}{F^2}\right)+ad\right]$$
$$>\frac{\alpha g_1g_2}{k\rho_2}+\frac{3\gamma(m_1+\rho_1)}{k^2\rho_2^4},\tag{47}$$
$$\left(1+\frac{fd}{F}\right)\left(\rho_2+m_2+\frac{\delta_1d^2}{F^2}+ad\right)<1,\tag{48}$$
$$\alpha\rho_1<\frac{12\gamma g_1^3(m_1+\rho_1)}{k\rho_2^4}.\tag{49}$$

Now, from Equation (32), we have

$$E_{2I}=\frac{[bE_{1p}+(\delta_2E_{1p}-m_3)(\alpha+\gamma E_{1p}^2)]}{F\left(\alpha+\gamma E_{1p}^2\right)}.\tag{50}$$

So, $E_7=\left(\dot{E}_{1r},\dot{E}_{1p},\dot{E}_{2s},\dot{E}_{2I}\right)$ exist if the next conditions hold, where $\dot{E}_{1p}=E_{1p}\left(\dot{E}_{1r}\right)$ *and* $\dot{E}_{2I}=E_{2I}(\dot{E}_{1p})$

$\delta_2\dot{E}_{1p} > m_3,$ (51)

$g_1(k - E_{1r}) < k(m_1 + \rho_1).$ (52)

LOCAL STABILITY (LS)

The local stability (LS) of system (1) have been discussed at each equilibrium point by obtaining the Jacobian matrix (JM), and determining the eigenvalues, of system (1), at all points.

Note that,, we denote the eigenvalues, of (JM) $J_i\,; i = 0, 1, \ldots 7$ that, characterize the dynamics in, the $E_{1r}, E_{1p}, E_{2s}, E_{2I}$ directions as $\lambda_{iE_{1r}}, \lambda_{iE_{1p}}, \lambda_{iE_{2s}}$ and $\lambda_{iE_{2I}}$ respectively. The Jacobian matrix $J(E_{1r}, E_{1p}, E_{2s}, E_{2I})$ of the system (1) at each point can be expressed as $J = [\Phi_{ij}]_{4\times4'}$.

$$\Phi_{11} = g_1\left(1 - \frac{2E_{1r}}{k}\right) - (\rho_1 + m_1), \qquad \Phi_{12} = \rho_2,$$

$$\Phi_{13} = \Phi_{14} = 0, \qquad \Phi_{21} = \rho_1,$$

$$\Phi_{22} = \frac{g_2}{1+fE_{2s}} - 2cE_{1p} - \frac{aE_{2s}(\alpha - \gamma E_{1p}^2)}{(\alpha + \gamma E_{1p}^2)^2} - \delta_1 E_{2s}^2 - (\rho_2 + m_2),$$

$$\Phi_{23} = \frac{-g_2 f E_{1p}}{(1+fE_{2s})^2} - \frac{aE_{1p}}{\alpha + \gamma E_{1p}^2} - 2\delta_1 E_{2s}E_{1p}, \qquad \Phi_{24} = 0,$$

$$\Phi_{31} = 0, \qquad \Phi_{32} = \frac{bE_{2s}(\alpha - \gamma E_{1p}^2)}{(\alpha + \gamma E_{1p}^2)^2} + \delta_2 E_{2s}^2,$$

$$\Phi_{33} = \frac{bE_{1p}}{\alpha + \gamma E_{1p}^2} + 2\delta_2 E_{1p}E_{2s} - FE_{2I} - m_3, \qquad \Phi_{34} = -FE_{2s},$$

$$\Phi_{41} = \Phi_{42} = 0, \qquad \Phi_{43} = FE_{2I}, \qquad \Phi_{44} = FE_{2s} - d.$$

- The (JM) of system (1) at $E_0 = (0, 0, 0,0)$ it can be written as,

$$J_0 = J(E_0) = \begin{bmatrix} g_1 - (\rho_1 + m_1) & \rho_2 & 0 & 0 \\ \rho_1 & g_2 - (\rho_2 + m_2) & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix} \quad (53)$$

So, the characteristic equation of J_0 is

$$(g_1 - (\rho_1 + m_1) - \lambda)(g_2 - (\rho_2 + m_2) - \lambda)(-m_3 - \lambda)(-\mathrm{d} - \lambda) = 0, \quad (54)$$

and hence the eigenvalues of J_0 are

$$\lambda_{0E_{1r}} = g_1 - (\rho_1 + m_1), \quad \lambda_{0E_{1p}} = g_2 - (\rho_2 + m_2), \quad \lambda_{0E_{2s}} = -m_3 < 0,$$

and $\lambda_{0E_{2I}} = -d < 0$.

Hence E_0 is a (LAS), by reversing conditions (2) and (3) and it is unstable, otherwise.

- The (JM) of system (1) at, $E_1 = (\bar{E}_{1r}, 0, 0,0)$ can be written as

$$J_1 = J(E_1) = \begin{bmatrix} g_1\left(1 - \frac{2\bar{E}_{1r}}{k}\right) - (\rho_1 + m_1) & \rho_2 & 0 & 0 \\ \rho_1 & g_2 - (\rho_2 + m_2) & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix} \quad (55)$$

So, the characteristic equation of J_1 is

$$\left(g_1\left(1 - \frac{2\bar{E}_{1r}}{k}\right) - (\rho_1 + m_1) - \lambda\right)(g_2 - (\rho_2 + m_2) - \lambda)[(-m_3 - \lambda)(-d - \lambda)] = 0, \quad (56)$$

and hence the eigenvalues of J_1 are:

$$\lambda_{1E_{1r}} = g_1\left(1 - \frac{2\bar{E}_{1r}}{k}\right) - (\rho_1 + m_1),$$

$$\lambda_{1E_{1p}} = g_2 - (\rho_2 + m_2),$$
$$\lambda_{1E_{2s}} = -m_3 < 0, \qquad and \qquad \lambda_{1E_{2I}} = -d < 0.$$

Hence E_1 is a locally asymptotically stable, by reversing condition (3) and in addition to the condition

$$\frac{k}{2} < \bar{E}_{1r} < k. \tag{57}$$

It is unstable otherwise.

- The (JM) of system (1) at $E_2 = \left(0\,, \bar{\bar{E}}_{1p}, 0{,}0\,\right)$ where $\bar{\bar{E}}_{1p} = \frac{g_2-(\rho_2+m_2)}{c}$ can be written as

$$J_2 = J(E_2) = \left[v_{ij}\,\right]_{4\times4} \tag{58}$$

where:

$$v_{11} = g_1 - (\rho_1 + m_1), \quad v_{12} = \rho_2\,, \quad v_{13} = 0, \quad v_{14} = 0,$$
$$v_{21} = \rho_1\,, \quad v_{22} = g_2 - 2c\bar{\bar{E}}_{1p} - (\rho_2 + m_2)\,,$$
$$v_{23} = -g_2 f\bar{\bar{E}}_{1p} - \frac{a\bar{\bar{E}}_{1p}}{\alpha+\gamma\bar{\bar{E}}_{1p}^2}, \quad v_{24} = 0\,, \quad v_{31} = 0\,, \quad v_{32} = 0,$$
$$v_{33} = \frac{b\bar{\bar{E}}_{1p}}{\alpha+\gamma\bar{\bar{E}}_{1p}^2} - m_3, \quad v_{34} = 0, \quad v_{41} = 0, \quad v_{42} = 0,$$
$$v_{43} = 0, \quad v_{44} = -d.$$

So, the characteristic equation of J_2 is:

$$(g_1 - (\rho_1 + m_1) - \lambda)(g_2 - 2c\bar{\bar{E}}_{1p} - (\rho_2 + m_2) - \lambda)\left[\left(\frac{b\bar{\bar{E}}_{1p}}{\alpha + \gamma\bar{\bar{E}}_{1p}^2} - m_3 - \lambda\right)(-d - \lambda)\right] = 0. \tag{59}$$

And hence the eigenvalues of J_2 are:

$$\lambda_{2E_{1r}} = g_1 - (\rho_1 + m_1), \quad \lambda_{2E_{1p}} = g_2 - 2c\bar{\bar{E}}_{1p} - (\rho_2 + m_2),$$
$$\lambda_{2E_{2s}} = \frac{b\bar{\bar{E}}_{1p}}{\alpha+\gamma\bar{\bar{E}}_{1p}^2} - m_3 \quad \text{and} \quad \lambda_{2E_{2I}} = -d.$$

Hence E_2 is locally asymptotically stable, by reversing condition (2) and in addition to the following conditions holds:

$$g_2 < 2c\bar{\bar{E}}_{1p} + (\rho_2 + m_2), \tag{60}$$
$$\frac{b\bar{\bar{E}}_{1p}}{\alpha + \gamma\bar{\bar{E}}_{1p}^2} < m_3. \tag{61}$$

It is unstable, otherwise.

- The (JM) of system (1) at, $E_3 = \left(E_{1r}^*, E_{1p}^*\,, 0{,}0\,\right)$ this can be written as

$$J_3 = J(E_3) = \left[\Theta^*{}_{ij}\,\right]_{4\times4} \tag{62}$$

where:

$$\Theta^*{}_{11} = g_1\left(1 - \frac{2E_{1r}^*}{k}\right) - (\rho_1 + m_1), \quad \Theta^*{}_{12} = \rho_2, \quad \Theta^*{}_{13} = \Theta^*{}_{14} = 0,$$
$$\Theta^*{}_{21} = \rho_1, \quad \Theta^*{}_{22} = g_2 - 2cE_{1p}^* - (\rho_2 + m_2), \quad \Theta^*{}_{23} = -g_2 fE_{1p}^* - \frac{aE_{1p}^*}{\alpha+\gamma E^*{}_{1p}^2}, \quad \Theta^*{}_{24} = 0,$$
$$\Theta^*{}_{31} = \Theta^*{}_{32} = 0, \quad \Theta^*{}_{33} = \frac{bE_{1p}^*}{\alpha+\gamma E^*{}_{1p}^2} - m_3, \quad \Theta^*{}_{34} = 0,$$
$$\Theta^*{}_{41} = \Theta^*{}_{42} = 0, \quad \Theta^*{}_{43} = 0, \quad \Theta^*{}_{44} = -\mathrm{d}.$$

So, the characteristic equation of J_3 is:

$$\left(g_1\left(1-\frac{2E_{1r}^*}{k}\right)-(\rho_1+m_1)-\lambda\right)\left(g_2-2cE_{1p}^*-(\rho_2+m_2)-\lambda\right)$$
$$\left[\left(\frac{bE_{1p}^*}{\alpha+\gamma {E^*}_{1p}^2}-m_3-\lambda\right)(-\mathrm{d}-\lambda)\right]=0. \quad (63)$$

And hence the eigenvalues of J_3 are:

$$\lambda_{3E_{1r}}=g_1\left(1-\frac{2E_{1r}^*}{k}\right)-(\rho_1+m_1), \quad \lambda_{3E_{1p}}=g_2-2cE_{1p}^*-(\rho_2+m_2),$$
$$\lambda_{3E_{2s}}=\frac{bE_{1p}^*}{\alpha+\gamma {E^*}_{1p}^2}-m_3, \quad \lambda_{3E_{2I}}=-d.$$

Hence E_3 is locally asymptotically stable under the following conditions:

$$E_{1r}^*>\frac{k}{2}, \quad (64)$$
$$g_2<2cE_{1p}^*+(\rho_2-m_2), \quad (65)$$
$$m_3>bE_{1p}^*/\alpha+\gamma {E^*}_{1p}^2. \quad (66)$$

It is unstable otherwise.

- The (JM) of system (1) at $E_4=\left(0,\tilde{E}_{1p},\tilde{E}_{2s},\tilde{E}_{2I}\right)$ can be written as

$$J_4=J(E_4)=\left[\widetilde{\omega}_{ij}^{[4]}\right]_{4\times 4} \quad (67)$$

where:

$$\widetilde{\omega}_{11}^{[4]}=g_1-(\rho_1+m_1), \quad \widetilde{\omega}_{12}^{[4]}=\rho_2, \quad \widetilde{\omega}_{13}^{[4]}=0, \quad \widetilde{\omega}_{14}^{[4]}=0,$$
$$\widetilde{\omega}_{21}^{[4]}=\rho_1, \quad \widetilde{\omega}_{22}^{[4]}=\frac{g_2}{1+f\tilde{E}_{2s}}-2c\tilde{E}_{1p}-\frac{a\tilde{E}_{2s}\left(\alpha-\gamma\tilde{E}_{1p}^2\right)}{(\alpha+\gamma\tilde{E}_{1p}^2)^2}-\delta_1\tilde{E}_{2s}^2-(\rho_2+m_2),$$
$$\widetilde{\omega}_{23}^{[4]}=\frac{-g_2f\tilde{E}_{1p}}{(1+f\tilde{E}_{2s})^2}-\frac{a\tilde{E}_{1p}}{\alpha+\gamma\tilde{E}_{1p}^2}-2\delta_1\tilde{E}_{2s}\tilde{E}_{1p}, \quad \widetilde{\omega}_{24}^{[4]}=0,$$
$$\widetilde{\omega}_{31}^{[4]}=0, \quad \widetilde{\omega}_{32}^{[4]}=\frac{b\tilde{E}_{2s}\left(\alpha-\gamma\tilde{E}_{1p}^2\right)}{(\alpha+\gamma\tilde{E}_{1p}^2)^2}+\delta_2\tilde{E}_{2s}^2,$$
$$\widetilde{\omega}_{33}^{[4]}=\frac{b\tilde{E}_{1p}}{\alpha+\gamma\tilde{E}_{1p}^2}+2\delta_2\tilde{E}_{1p}\tilde{E}_{2s}-F\tilde{E}_{2I}-m_3,$$
$$\widetilde{\omega}_{34}^{[4]}=-F\tilde{E}_{2s}, \quad \widetilde{\omega}_{41}^{[4]}=\widetilde{\omega}_{42}^{[4]}=0, \quad \widetilde{\omega}_{43}^{[4]}=F\tilde{E}_{2I}, \quad \widetilde{\omega}_{44}^{[4]}=F\tilde{E}_{2s}-d.$$

Finally, the characteristic equation of J_4 is given by,:

$$\lambda^4+\widetilde{\mathcal{B}}_1^{[4]}\lambda^3+\widetilde{\mathcal{B}}_2^{[4]}\lambda^2+\widetilde{\mathcal{B}}_3^{[4]}\lambda+\widetilde{\mathcal{B}}_4^{[4]}=0, \quad (68)$$

where

$$\widetilde{\mathcal{B}}_1^{[4]}=-\left[\widetilde{\omega}_{11}^{[4]}+\widetilde{\omega}_{33}^{[4]}+\widetilde{\omega}_{44}^{[4]}\right],$$

$$\widetilde{\mathcal{B}}_2^{[4]}=\left[\left(\widetilde{\omega}_{33}^{[4]}+\widetilde{\omega}_{44}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]}+\widetilde{\omega}_{22}^{[4]}\right)+\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}-\left(\widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}+\widetilde{\omega}_{23}^{[4]}\widetilde{\omega}_{32}^{[4]}+\widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}\right)\right],$$

$$\widetilde{\mathcal{B}}_3^{[4]}=-\left[\left(\widetilde{\omega}_{33}^{[4]}+\widetilde{\omega}_{44}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{22}^{[4]}-\widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}\right)+\left(\widetilde{\omega}_{11}^{[4]}+\widetilde{\omega}_{22}^{[4]}\right)\left(\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}-\widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}\right)-\right.$$
$$\left.\widetilde{\omega}_{23}^{[4]}\widetilde{\omega}_{32}^{[4]}\left(\widetilde{\omega}_{11}^{[4]}+\widetilde{\omega}_{44}^{[4]}\right)\right],$$

$$\widetilde{\mathcal{B}}_4^{[4]}=\left[\left(\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}-\widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{22}^{[4]}-\widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}\right)-\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{23}^{[4]}\widetilde{\omega}_{32}^{[4]}\widetilde{\omega}_{44}^{[4]}\right].$$

So, by Routh-Hurwitz criteria, Equation (68) has real negative real parts, if and only if $\widetilde{\mathcal{B}}_1^{[4]} > 0, \widetilde{\mathcal{B}}_3^{[4]} > 0, \widetilde{\mathcal{B}}_4^{[4]} > 0$ and $\tilde{\Delta} = \left(\widetilde{\mathcal{B}}_1^{[4]}\widetilde{\mathcal{B}}_2^{[4]} - \widetilde{\mathcal{B}}_3^{[4]}\right)\widetilde{\mathcal{B}}_3^{[4]} - \widetilde{\mathcal{B}}^2{}_1^{[4]}\widetilde{\mathcal{B}}_4^{[4]} > 0.$

Clearly,, we have $\widetilde{\mathcal{B}}_i^{[4]} > 0, i = 1,3,4$ if in addition to reversing conditions,(2) the following conditions hold,::

$$\tilde{E}_{1p}^2 < \frac{\alpha}{\gamma}, \tag{69}$$

$$\frac{g_2}{1 + f\tilde{E}_{2s}} < 2c\tilde{E}_{1p} + \frac{a\tilde{E}_{2s}(\alpha - \gamma\tilde{E}_{1p}^2)}{(\alpha + \gamma\tilde{E}_{1p}^2)^2} + \delta_1\tilde{E}_{2s}^2 + (\rho_2 + m_2), \tag{70}$$

$$\frac{b\tilde{E}_{1p}}{\alpha + \gamma\tilde{E}_{1p}^2} + 2\delta_2\tilde{E}_{1p}\tilde{E}_{2s} < F\tilde{E}_{2I} + m_3, \tag{71}$$

$$\tilde{E}_{2s} < \frac{F}{d}, \tag{72}$$

$$\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{22}^{[4]} > \widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}. \tag{73}$$

Straightforward computation shows that:

$\tilde{\Delta} = \tilde{\Delta}_1 - \tilde{\Delta}_2$, ,where:

$$\tilde{\Delta}_1 = \left[\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]} - \widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}\left(2\widetilde{\omega}_{11}^{[4]} + \widetilde{\omega}_{22}^{[4]}\right) + \widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}\left(\widetilde{\omega}_{33}^{[4]} + \widetilde{\omega}_{44}^{[4]} - \widetilde{\omega}_{22}^{[4]}\right)\right]\left[\left(\widetilde{\omega}_{33}^{[4]} + \widetilde{\omega}_{44}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{22}^{[4]} - \widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}\right)\right.$$
$$\left. + \left(\widetilde{\omega}_{11}^{[4]} + \widetilde{\omega}_{22}^{[4]}\right)\left(\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]} - \widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}\right)\right],$$

$$\tilde{\Delta}_2 = \left[\widetilde{\omega}^2{}_{11}^{[4]} + \left(\widetilde{\omega}_{33}^{[4]} + \widetilde{\omega}_{44}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]} + \widetilde{\omega}_{22}^{[4]}\right) + \widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}\right]\left(\widetilde{\omega}_{33}^{[4]} + \widetilde{\omega}_{44}^{[4]}\right) + \widetilde{\omega}_{22}^{[4]}\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]}][\widetilde{\omega}_{23}^{[4]}\widetilde{\omega}_{32}^{[4]}\left(\widetilde{\omega}_{11}^{[4]} + \widetilde{\omega}_{44}^{[4]}\right) + (\widetilde{\omega}_{11}^{[4]}$$
$$+ \widetilde{\omega}_{33}^{[4]} + \widetilde{\omega}_{44}^{[4]})^2\left[\left(\widetilde{\omega}_{33}^{[4]}\widetilde{\omega}_{44}^{[4]} - \widetilde{\omega}_{34}^{[4]}\widetilde{\omega}_{43}^{[4]}\right)\left(\widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{22}^{[4]} - \widetilde{\omega}_{12}^{[4]}\widetilde{\omega}_{21}^{[4]}\right) - \widetilde{\omega}_{11}^{[4]}\widetilde{\omega}_{23}^{[4]}\widetilde{\omega}_{32}^{[4]}\widetilde{\omega}_{44}^{[4]}\right],$$

Consequently, $\tilde{\Delta} = \left(\widetilde{\mathcal{B}}_1^{[4]}\widetilde{\mathcal{B}}_2^{[4]} - \widetilde{\mathcal{B}}_3^{[4]}\right)\widetilde{\mathcal{B}}_3^{[4]} - \widetilde{\mathcal{B}}^2{}_1^{[4]}\widetilde{\mathcal{B}}_4^{[4]} > 0$ if in addition of the above conditions (69 − 73) the following conditions hold,:

$$\tilde{\Delta}_1 > \tilde{\Delta}_2 \tag{74}$$

So, eigenvalues of J_4 have a negative real part, which indicates that E_4 is locally stable, and unstable otherwise.

- The (JM) of system (1) of $E_5 = \left(\breve{E}_{1r}, \breve{E}_{1p}, \breve{E}_{2s}, 0\right)$ can be written as,

$$J_5 = J(E_5) = \left[\,\breve{\mathcal{L}}_{ij}\right]_{4\times4}, \tag{75}$$

where:

$$\breve{\mathcal{L}}_{11} = g_1\left(1 - \tfrac{2\breve{E}_{1r}}{k}\right) - (\rho_1 + m_1), \qquad \breve{\mathcal{L}}_{12} = \rho_2, \qquad \breve{\mathcal{L}}_{13} = \breve{\mathcal{L}}_{14} = 0,$$

$$\breve{\mathcal{L}}_{21} = \rho_1, \qquad \breve{\mathcal{L}}_{22} = \tfrac{g_2}{1 + f\breve{E}_{2s}} - 2c\hat{E}_{1p} - \tfrac{a\breve{E}_{2s}(\alpha - \gamma\breve{E}_{1p}^2)}{(\alpha + \gamma\breve{E}_{1p}^2)^2} - \delta_1\breve{E}_{2s}^2 - (\rho_2 + m_2),$$

$$\breve{\mathcal{L}}_{23} = \tfrac{-g_2 f\breve{E}_{1p}}{(1 + f\breve{E}_{2s})^2} - \tfrac{a\breve{E}_{1p}}{\alpha + \gamma\breve{E}_{1p}^2} - 2\delta_1\breve{E}_{2s}\breve{E}_{1p}, \qquad \breve{\mathcal{L}}_{24} = \breve{\mathcal{L}}_{31} = 0,$$

$$\breve{\mathcal{L}}_{32} = \frac{b\breve{E}_{2s}(\alpha - \gamma\breve{E}_{1p}^2)}{(\alpha + \gamma\breve{E}_{1p}^2)^2} + \delta_2\breve{E}_{2s}^2, \qquad \breve{\mathcal{L}}_{33} = \frac{b\breve{E}_{1p}}{\alpha + \gamma\breve{E}_{1p}^2} + 2\delta_2\breve{E}_{1p}\breve{E}_{2s} - m_3,$$

$$\breve{\mathcal{L}}_{34} = -F\breve{E}_{2s}, \qquad \breve{\mathcal{L}}_{41} = \breve{\mathcal{L}}_{42} = \breve{\mathcal{L}}_{43} = 0, \qquad \breve{\mathcal{L}}_{44} = F\breve{E}_{2s} - d.$$

Finally, the characteristic equation of J_5 is given by,

$$\lambda^4 + \breve{\mathfrak{B}}_1\lambda^3 + \breve{\mathfrak{B}}_2\,\lambda^2 + \breve{\mathfrak{B}}_3\lambda + \breve{\mathfrak{B}}_4 = 0, \tag{76}$$

where

$$\breve{\mathfrak{B}}_1 = -[\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{22} + \check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44}],$$
$$\breve{\mathfrak{B}}_2 = [\check{\mathcal{L}}_{11}(\check{\mathcal{L}}_{22} + \check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44}) + \check{\mathcal{L}}_{22}(\check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44}) + \check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44} - \check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32} - \check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21}],$$
$$\breve{\mathfrak{B}}_3 = -[(\check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44})(\check{\mathcal{L}}_{11}\check{\mathcal{L}}_{22} - \check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21}) + (\check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44})(\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{22}) - \check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32}(\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{44})],$$
$$\breve{\mathfrak{B}}_4 = [\check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44}(\check{\mathcal{L}}_{11}\check{\mathcal{L}}_{22} - \check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21}) - \check{\mathcal{L}}_{11}\check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32}\check{\mathcal{L}}_{44}].$$

So, by Routh-Hurwitz criteria, Equation (76) has real negative real parts, if and only if $\breve{\mathfrak{B}}_1 > 0, \breve{\mathfrak{B}}_3 > 0, \breve{\mathfrak{B}}_4 > 0$ and $\breve{\Delta} = (\breve{\mathfrak{B}}_1\breve{\mathfrak{B}}_2 - \breve{\mathfrak{B}}_3\,)\breve{\mathfrak{B}}_3 \, - \breve{\mathfrak{B}}_1^2\breve{\mathfrak{B}}_4 \, > 0.$

Clearly,, we have $\breve{\mathfrak{B}}_i > 0, i = 1,3,4$ if in addition to reversing conditions (2) the following conditions are hold::

$$\breve{E}_{1r} > \frac{k}{2},, \tag{77}$$

$$\breve{E}_{1p}^2 < \frac{\alpha}{\gamma}, \tag{78}$$

$$\frac{g_2}{1 + f\hat{E}_{2s}} < 2c\breve{E}_{1p} + \frac{a\breve{E}_{2s}(\alpha - \gamma\breve{E}_{1p}^2)}{(\alpha + \gamma\breve{E}_{1p}^2)^2} + \delta_1\breve{E}_{2s}^2 + (\rho_2 + m_2), \tag{79}$$

$$\frac{b\breve{\mathrm{E}}_{1p}}{\alpha + \gamma\breve{\mathrm{E}}_{1p}^2} + 2\delta_2\breve{\mathrm{E}}_{1p}\breve{\mathrm{E}}_{2s} < m_3, \tag{80}$$

$$\breve{\mathrm{E}}_{2s} < \frac{F}{d}, \tag{81}$$

$$\check{\mathcal{L}}_{11}\check{\mathcal{L}}_{22} > \check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21}. \tag{82}$$

Straightforward computation shows that:

$\breve{\Delta} = \breve{\Delta}_1 - \breve{\Delta}_2$,where:

$$\breve{\Delta}_1 = [\check{\mathcal{L}}_{11}(\check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21} - \check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44}) + (\check{\mathcal{L}}_{22} + \check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44})[-\check{\mathcal{L}}_{11}(\check{\mathcal{L}}_{22} + \check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44}) - \check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44}] + \check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44}(\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{22}) - \check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32}(\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{44})][(\check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44})(\check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21} - \check{\mathcal{L}}_{11}\check{\mathcal{L}}_{22}) + \check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32}(\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{44})],$$

$$\breve{\Delta}_2 = \check{\mathcal{L}}_{11}^2 + \check{\mathcal{L}}_{22}(\check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44})[\check{\mathcal{L}}_{22}(\check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44})] + [\check{\mathcal{L}}_{11} + \check{\mathcal{L}}_{22} + \check{\mathcal{L}}_{33} + \check{\mathcal{L}}_{44}]^2\check{\mathcal{L}}_{33}\check{\mathcal{L}}_{44}(\check{\mathcal{L}}_{11}\check{\mathcal{L}}_{22} - \check{\mathcal{L}}_{12}\check{\mathcal{L}}_{21}) + \check{\mathcal{L}}_{11}\check{\mathcal{L}}_{23}\check{\mathcal{L}}_{32}\check{\mathcal{L}}_{44},$$

Consequently, $\breve{\Delta} = (\breve{\mathfrak{B}}_1\breve{\mathfrak{B}}_2 - \breve{\mathfrak{B}}_3\,)\breve{\mathfrak{B}}_3 \, - \breve{\mathfrak{B}}_1^2\breve{\mathfrak{B}}_4 \, > 0$ if in addition of the above conditions (77 − 82) the following conditions hold,:

$$\breve{\Delta}_1 > \breve{\Delta}_2 \tag{83}$$

So, eigenvalues of J_5 have a negative real part, which indicates that E_5 is locally stable . Similarly for (EP) $E_6 = (\breve{\mathrm{E}}'_{1r}, \breve{\mathrm{E}}'_{1p}, \breve{\mathrm{E}}'_{2s}, 0)$.

Otherwise, E_5 and E_6 are unstable.

- Finally, the (JM) of system (1) of $E_7 = (\dot{E}_{1r}, \dot{E}_{1p}, \dot{E}_{2s}, \dot{E}_{2I})$ it can be written as:

$$J_7 = J(E_7) = [\grave{\mathscr{b}}_{ij}\,]_{4\times 4} \tag{84}$$

where:

$$\grave{\mathscr{b}}_{11} = g_1\left(1 - \frac{2\dot{E}_{1r}}{k}\right) - (\rho_1 + m_1), \quad \grave{\mathscr{b}}_{12} = \rho_2, \quad \grave{\mathscr{b}}_{13} = \grave{\mathscr{b}}_{14} = 0,$$

$$\grave{\mathscr{b}}_{21} = \rho_1, \quad \grave{\mathscr{b}}_{22} = \frac{g_2}{1 + f\dot{E}_{2s}} - 2c\dot{E}_{1p} - \frac{aE_{2s}(\alpha - \gamma\dot{E}_{1p}^2)}{(\alpha + \gamma\dot{E}_{1p}^2)^2} - \delta_1\dot{E}_{2s}^2 - (\rho_2 + m_2),$$

$$\grave{b}_{23} = \frac{-g_2 f \dot{E}_{1p}}{(1+f\dot{E}_{2s})^2} - \frac{a\dot{E}_{1p}}{\alpha+\gamma\dot{E}_{1p}^2} - 2\delta_1 \dot{E}_{2s}\dot{E}_{1p}, \quad \grave{b}_{24} = 0,$$

$$\grave{b}_{31} = 0, \quad \grave{b}_{32} = \frac{b\dot{E}_{2s}(\alpha-\gamma\dot{E}_{1p}^2)}{(\alpha+\gamma\dot{E}_{1p}^2)^2} + \delta_2\dot{E}_{2s}^2, \quad \grave{b}_{33} = \frac{b\dot{E}_{1p}}{\alpha+\gamma\dot{E}_{1p}^2} + 2\delta_2\dot{E}_{1p}\dot{E}_{2s} - F\dot{E}_{2I} - m_3,$$

$$\grave{b}_{34} = -F\dot{E}_{2s}, \quad \grave{b}_{41} = \grave{b}_{42} = 0, \quad \grave{b}_{43} = F\dot{E}_{2I}, \quad \grave{b}_{44} = F\dot{E}_{2s} - d.$$

Finally, the characteristic equation of J_7 is given by,

$$\lambda^4 + \dot{\beta}_1\lambda^3 + \dot{\beta}_2\lambda^2 + \dot{\beta}_3\lambda + \dot{\beta}_4 = 0, \tag{85}$$

where

$$\dot{\beta}_1 = -[\grave{b}_{11} + \grave{b}_{33} + \grave{b}_{44}],$$
$$\dot{\beta}_2 = [(\grave{b}_{33}+\grave{b}_{44})(\grave{b}_{11}+\grave{b}_{22}) + \grave{b}_{33}\grave{b}_{44} - (\grave{b}_{34}\grave{b}_{43} + \grave{b}_{23}\grave{b}_{32} + \grave{b}_{12}\grave{b}_{21})],$$
$$\dot{\beta}_3 = -[(\grave{b}_{33}+\grave{b}_{44})(\grave{b}_{11}\grave{b}_{22} - \grave{b}_{12}\grave{b}_{21}) + (\grave{b}_{11}+\grave{b}_{22})(\grave{b}_{33}\grave{b}_{44} - \grave{b}_{34}\grave{b}_{43}) - \grave{b}_{23}\grave{b}_{32}(\grave{b}_{11}+\grave{b}_{44})],$$
$$\dot{\beta}_4 = [(\grave{b}_{33}\grave{b}_{44} - \grave{b}_{34}\grave{b}_{43})(\grave{b}_{11}\grave{b}_{22} - \grave{b}_{12}\grave{b}_{21}) - \grave{b}_{11}\grave{b}_{23}\grave{b}_{32}\grave{b}_{44}].$$

So, by Routh-Hurwitz criteria, Equation (85) has real negative real parts, if and only if $\dot{\beta}_1 > 0, \dot{\beta}_3 > 0, \dot{\beta}_4 > 0$ and $\dot{\Delta} = (\dot{\beta}_1\dot{\beta}_2 - \dot{\beta}_3)\dot{\beta}_3 - \dot{\beta}_1^2\dot{\beta}_4 > 0.$

Clearly,, we have $\dot{\beta}_i > 0, i = 1,3,4$ if in addition to reversing condition,(2) the following conditions are hold::

$$\dot{E}_{1r} < \frac{k}{2}, \tag{86}$$

$$\dot{E}_{1p}^2 < \frac{\alpha}{\gamma}, \tag{87}$$

$$\frac{g_2}{1+f\dot{E}_{2s}} < 2c\dot{E}_{1p} + \frac{aE_{2s}(\alpha-\gamma\dot{E}_{1p}^2)}{(\alpha+\gamma\dot{E}_{1p}^2)^2} + \delta_1\dot{E}_{2s}^2 + (\rho_2+m_2), \tag{88}$$

$$F\dot{E}_{2I} + m_3 > \frac{b\dot{E}_{1p}}{\alpha+\gamma\dot{E}_{1p}^2} + 2\delta_2\dot{E}_{1p}\dot{E}_{2s}, \tag{89}$$

$$\dot{E}_{2s} < \frac{F}{d}, \tag{90}$$

$$\grave{b}_{11}\grave{b}_{22} > \grave{b}_{12}\grave{b}_{21}. \tag{91}$$

Straightforward computation shows that:

$\dot{\Delta} = \dot{\Delta}_1 - \dot{\Delta}_2$, where:

$$\dot{\Delta}_1 = [\grave{b}_{11}\grave{b}_{12}\grave{b}_{21} - \grave{b}_{33}\grave{b}_{44}(2\grave{b}_{11}+\grave{b}_{22}) + \grave{b}_{34}\grave{b}_{43}(\grave{b}_{33}+\grave{b}_{44}-\grave{b}_{22})][(\grave{b}_{33}+\grave{b}_{44})(\grave{b}_{11}\grave{b}_{22}-\grave{b}_{12}\grave{b}_{21}) + (\grave{b}_{11}+\grave{b}_{22})(\grave{b}_{33}\grave{b}_{44}-\grave{b}_{34}\grave{b}_{43})],$$

$$\dot{\Delta}_2 = [\grave{b}_{11}^2 + (\grave{b}_{33}+\grave{b}_{44})(\grave{b}_{11}+\grave{b}_{22}) + \grave{b}_{33}\grave{b}_{44}](\grave{b}_{33}+\grave{b}_{44}) + \grave{b}_{22}\grave{b}_{33}\grave{b}_{44}][\grave{b}_{23}\grave{b}_{32}(\grave{b}_{11}+\grave{b}_{44}) + (\grave{b}_{11}+\grave{b}_{33}+\grave{b}_{44})^2[(\grave{b}_{33}\grave{b}_{44}-\grave{b}_{34}\grave{b}_{43})(\grave{b}_{11}\grave{b}_{22}-\grave{b}_{12}\grave{b}_{21}) - \grave{b}_{11}\grave{b}_{23}\grave{b}_{32}\grave{b}_{44}],$$

Consequently, $\dot{\Delta} = (\dot{\beta}_1\dot{\beta}_2 - \dot{\beta}_3)\dot{\beta}_3 - \dot{\beta}_1^2\dot{\beta}_4 > 0$ if in addition of the above conditions $(86-91)$ the following conditions hold,:

$$\dot{\Delta}_1 > \dot{\Delta}_2 \tag{92}$$

So, eigenvalues of J_7 have a negative real part, which indicates that E_7 is locally stable, and unstable otherwise

GLOBAL STABILITY (GS)

In this section the (GS) of the (Eps) of system (1), which are (LAS) of system (1), have been analytically, studied with the suitable Lyapunov functions.

Theorem 5.1: The (EP) $E_0 = (0,0,0,0)$ is a globally asymptotically stable (GAS) in the subregion $\phi_0 \subset R_+^4$ that satisfies the next condition:

$$m_1E_{1r} + m_2E_{1p} + m_3E_{2s} + dE_{2I} > \frac{1}{4}\left(kg_1 + \frac{g_2^2}{c}\right), \tag{93}$$

Proof: Regard the function,:

$\dot{Z}_0(E_{1r}, E_{1p}, E_{2s}, E_{2I}) = E_{1r} + E_{1p} + E_{2s} + E_{2I}$.

Clearly, $\dot{Z}_0\colon \mathrm{R}_+^4 \to \mathrm{R}$ where $\dot{Z}_0 \in \mathrm{C}^1$ is positive definite function,

Now, by deriving $\dot{Z}_0$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_0}{dt} < g_1E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - m_1E_{1r} + \frac{g_2E_{1p}}{1+fE_{2s}} - cE_{1p}^2 - (a-b)\frac{E_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} - (\delta_1-\delta_2)E_{1p}E_{2s}^2 - m_2E_{1p} - m_3E_{2s} - dE_{2I},$$

Hence, by the biological facts $b < a$ and $\delta_2 < \delta_1$, so

$$\frac{d\dot{Z}_0}{dt} < g_1E_{1r}\left(1 - \frac{E_{1r}}{k}\right) - m_1E_{1r} + \frac{g_2E_{1p}}{1+fE_{2s}} - cE_{1p}^2 - m_2E_{1p} - m_3E_{2s} - dE_{2I},$$

Now, since the logistic functions $f_1(E_{1r}) = g_1E_{1r} - \frac{g_1E_{1r}^2}{k}$ and $f_2(E_{1p}) = g_2E_{1p}\left(1 - \frac{c}{g_2}E_{1p}\right)$ are bounded by constants $\left(\frac{kg_1}{4}\right), \left(\frac{g_2^2}{4c}\right)$ respectively. So

$$\frac{d\dot{Z}_0}{dt} < \frac{kg_1}{4} + \frac{g_2^2}{4c} - [m_1E_{1r} + m_2E_{1p} + m_3E_{2s} + dE_{2I}],$$

So, by condition (93) $\frac{d\dot{Z}_0}{dt} < 0$, and hence $\dot{Z}_0$ is a Lyapunov function which gives a global asymptotic of E_0 .

Theorem 5.2: The (EP) $E_1 = (\bar{E}_{1r}, 0, 0,0)$ is a (GAS) in the subregion $\phi_1 \subset R_+^4$ that satisfies the next conditions:

$$E_{1r} > \bar{E}_{1r}, \tag{94}$$

$$\bar{N}_{\acute{1}} > \bar{N}_{\acute{2}}, \tag{95}$$

where:

$$\bar{N}_{\acute{1}} = m_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + m_2E_{1p} + m_3E_{2s} + dE_{2I},$$

$$\bar{N}_{\acute{2}} = g_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + \frac{g_2^2}{4c}.$$

Proof: Regard the function:

$$\dot{Z}_1(E_{1r}, E_{1p}, E_{2s}, E_{2I}) = \left(E_{1r} - \bar{E}_{1r} - \bar{E}_{1r}\ln\frac{E_{1r}}{\bar{E}_{1r}}\right) + E_{1p} + E_{2s} + E_{2I}.$$

Clearly, $\dot{Z}_1\colon \mathrm{R}_+^4 \to \mathrm{R}$ where $\dot{Z}_1 \in \mathrm{C}^1$ is positive definite function.

Now, by deriving $\dot{Z}_1$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_1}{dt} \le g_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} - \frac{g_1}{k}(E_{1r}-\bar{E}_{1r})(E_{1r}+\bar{E}_{1r}) - m_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + \frac{g_2E_{1p}}{1+fE_{2s}} - cE_{1p}^2 - (a-b)\frac{E_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} - (\delta_1-\delta_2)E_{1p}E_{2s}^2 - m_2E_{1p} - m_3E_{2s} - dE_{2I},$$

by the biological facts, $b < a$ & $\delta_2 < \delta_1$ and condition (94), so

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$$\frac{d\dot{Z}_1}{dt} < g_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} - \frac{g_1}{k}\frac{(E_{1r}-\bar{E}_{1r})^2(E_{1r}+\bar{E}_{1r})}{E_{1r}} - \frac{g_1}{k}(E_{1r}-\bar{E}_{1r})(E_{1r}+\bar{E}_{1r}) - m_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + \frac{g_2E_{1p}}{1+fE_{2s}} - cE_{1p}^2 - m_2E_{1p} - m_3E_{2s} - dE_{2I},$$

Now, since the logistic function $f(E_{1p}) = g_2E_{1p}\left(1-\frac{c}{g_2}E_{1p}\right)$ is bounded by the constant $\left(\frac{g_2^2}{4c}\right)$. So

$$\frac{d\dot{Z}_1}{dt} < -\left[m_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + m_2E_{1p} + m_3E_{2s} + dE_{2I}\right] + g_1\frac{(E_{1r}-\bar{E}_{1r})^2}{E_{1r}} + \frac{g_2^2}{4c},$$

$\frac{d\dot{Z}_1}{dt} < -\bar{N}_1 + \bar{N}_2$, where $\bar{N}_1$ and $\bar{N}_2$ are provided in the state of theorem.

So, by condition (95) $\frac{d\dot{Z}_1}{dt} < 0$,and hence $\dot{Z}_1$ is a Lyapunov function which gives a global asymptotic of E_1.

Theorem 5.3: The (EP) $E_2 = (0, \bar{\bar{E}}_{1p}, 0, 0)$ is a (GAS) in the subregion $\Phi_2 \subset R_+^4$ that satisfies the next conditions:

$$E_{1p} > \bar{\bar{E}}_{1p}, \quad (96)$$

$$\bar{\bar{N}}_1 > \bar{\bar{N}}_2, \quad (97)$$

where

$$\bar{\bar{N}}_1 = \frac{m_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 + m_1E_{1r} + c\frac{(E_{1p}-\bar{\bar{E}}_{1p})^2(E_{1p}+\bar{\bar{E}}_{1p})}{E_{1p}} + m_3E_{2s} + dE_{2I},$$

$$\bar{\bar{N}}_2 = \frac{g_1k}{4} + \frac{g_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 + \frac{a\bar{\bar{E}}_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\bar{\bar{E}}_{1p}E_{2s}^2.$$

Proof: Regard the, function :

$$\dot{Z}_2(E_{1r}, E_{1p}, E_{2s}, E_{2I}) = E_{1r} + \left(E_{1p} - \bar{\bar{E}}_{1p} - \bar{\bar{E}}_{1p}ln\frac{E_{1p}}{\bar{\bar{E}}_{1p}}\right) + E_{2s} + E_{2I}.$$

Clearly, $\dot{Z}_2: \mathrm{R}_+^4 \to \mathrm{R}$ where $\dot{Z}_2 \in \mathrm{C}^1$ is a positive definite function.

Now, by deriving $\dot{Z}_2$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_2}{dt} \le g_1E_{1r}\left(1-\frac{E_{1r}}{k}\right) - m_1E_{1r} + \frac{g_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 - c\frac{(E_{1p}-\bar{\bar{E}}_{1p})^2(E_{1p}+\bar{\bar{E}}_{1p})}{E_{1p}} + \frac{a\bar{\bar{E}}_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\bar{\bar{E}}_{1p}E_{2s}^2 - (a - b)\frac{E_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} - (\delta_1-\delta_2)E_{1p}E_{2s}^2 - \frac{m_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 - m_3E_{2s} - dE_{2I},$$

by the biological facts , $b < a$ & $\delta_2 < \delta_1$, and condition (96),so

$$\frac{d\dot{z}_2}{dt} < g_1E_{1r}\left(1-\frac{E_{1r}}{k}\right) + \frac{g_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 + \frac{a\bar{\bar{E}}_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\bar{\bar{E}}_{1p}E_{2s}^2 - c\frac{(E_{1p}-\bar{\bar{E}}_{1p})^2(E_{1p}+\bar{\bar{E}}_{1p})}{E_{1p}} - \frac{m_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 - m_1E_{1r} - m_3E_{2s} - dE_{2I},$$

Now, since the logistic function $f(E_{1p}) = g_1E_{1r}\left(1-\frac{E_{1r}}{k}\right)$ is bounded by the constant $\left(\frac{kg_1}{4}\right)$. So

$$\frac{d\dot{z}_2}{dt} < -\left[\frac{m_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 + m_1E_{1r} + m_3E_{2s} + dE_{2I}\right] + \frac{g_1k}{4} + \frac{g_2}{E_{1p}}(E_{1p}-\bar{\bar{E}}_{1p})^2 + \frac{a\bar{\bar{E}}_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\bar{\bar{E}}_{1p}E_{2s}^2,$$

$\frac{d\dot{Z}_2}{dt} < -\bar{\bar{N}}_1 + \bar{\bar{N}}_2$, where $\bar{\bar{N}}_1$ and $\bar{\bar{N}}_2$ are provided in the statement of the theorem.

So, by condition(97) $\frac{d\dot{Z}_2}{dt} < 0$, and hence $\dot{Z}_2$ is a Lyapunov function gives global asymptotic of E_2.

Theorem 5.4: The (EP) $E_3 = \left(E_{1r}^*, E_{1p}^*, 0,0\right)$ is a (GAS) in the subregion $\varphi_3 \subset R_+^4$ that satisfies the next conditions,:

$$E_{1r} > E_{1r}^*, E_{1p} > E_{1p}^*, \tag{98}$$

$$\dot{N}_1 > \dot{N}_2, \tag{99}$$

Where:

$$\dot{N}_1 = m_1\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) + m_3E_{2s} + dE_{2I},$$

$$\dot{N}_2 = \left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right)\frac{kg_1}{4} + \rho_1E_{1r}^*\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) + \rho_2E_{1r}^*E_{1p}^* + \delta_1E_{1p}^*E_{2s}^2 + \frac{aE_{1p}^*E_{2s}}{\alpha+\gamma E_{1p}^2}.$$

Proof: Consider the following function

$$\dot{Z}_3(E_{1r}, E_{1p}, E_{2s}, E_{2I}) = \left(E_{1r} - E_{1r}^* - E_{1r}^* ln\frac{E_{1r}}{E_{1r}^*}\right) + \left(E_{1p} - E_{1p}^* - E_{1p}^* ln\frac{E_{1p}}{E_{1p}^*}\right) + E_{2s} + E_{2I}$$

Clearly, $\dot{Z}_3: \mathrm{R}_+^4 \to \mathrm{R}$ where $\dot{Z}_3 \in \mathrm{C}^1$ is a positive definite function.

Now, by deriving $\dot{Z}_3$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_3}{dt} \le \left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right)\frac{g_1k}{4} + \rho_1E_{1r}^*\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) + \rho_2E_{1r}^*E_{1p}^* + \frac{g_2}{1+fE_{2s}}(E_{1p} - E_{1p}^*) + \delta_1E_{1p}^*E_{2s}^2 + \frac{aE_{1p}^*E_{2s}}{\alpha+\gamma E_{1p}^2} +$$
$$\frac{\rho_1E_{1r}^*E_{1p}^*}{E_{1p}} + \rho_2E_{1p}^*\left(\frac{E_{1p}-E_{1p}^*}{E_{1p}}\right) - \frac{\rho_2E_{1r}^*E_{1p}}{E_{1r}} - m_1\frac{(E_{1r}-E_{1r}^*)^2}{E_{1r}} - \left[g_1E_{1r}^*\left(1-\frac{E_{1r}^*}{k}\right)\right]\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) -$$
$$c\frac{(E_{1p}-E_{1p}^*)^2(E_{1p}+E_{1p}^*)}{E_{1p}} - \frac{\rho_1E_{1r}E_{1p}^*}{E_{1p}} - (a-b)\frac{E_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2} - (\delta_1-\delta_2)E_{1p}E_{2s}^2 - \frac{m_2}{E_{1p}}(E_{1p} - \bar{\bar{E}}_{1p})^2 -$$
$$g_2E_{1p}^*\left(\frac{E_{1p}-E_{1p}^*}{E_{1p}}\right) - m_3E_{2s} - dE_{2I},$$

by the biological facts $b < a$ & $\delta_2 < \delta_1$ and condition (98),so

$$\frac{d\dot{Z}_3}{dt} < -\left[m_1\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) + m_3E_{2s} + dE_{2I}\right] + \left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right)\frac{g_1k}{4} + \rho_1E_{1r}^*\left(\frac{E_{1r}-E_{1r}^*}{E_{1r}}\right) + \rho_2E_{1r}^*E_{1p}^* +$$
$$\delta_1E_{1p}^*E_{2s}^2 + \frac{aE_{1p}^*E_{2s}}{\alpha+\gamma E_{1p}^2},$$

$\frac{d\dot{Z}_3}{dt} < -\dot{N}_1 + \dot{N}_2$,, where $\dot{N}_1$ and $\dot{N}_2$ are provided in the state of theorem.

So, by condition(99) $\frac{d\dot{Z}_3}{dt} < 0$, and hence $\dot{Z}_3$ is a Lyapunov function that gives global asymptotic of E_3.

Theorem 5.5: The point $E_4 = \left(0, \tilde{E}_{1p}, \tilde{E}_{2s}, \tilde{E}_{2I}\right)$ is a (GAS) in the subregion $\phi_4 \subset R_+^4$ that satisfies the next conditions,:

$$E_{1p} > \tilde{E}_{1p}, E_{2s} > \tilde{E}_{2s}, E_{2I} > \tilde{E}_{2I}, \tag{100}$$

$$\tilde{N}_1 > \tilde{N}_2, \tag{101}$$

where:

$$\tilde{N}_1 = g_1E_{1r}\left(\frac{E_{1r}}{k} - 1\right) + (m_1+\rho_1)E_{1r} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)E_{1p}(cE_{1p}+\rho_2) + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(m_2E_{1p} + \frac{g_2\tilde{E}_{1p}}{1+f\tilde{E}_{2s}}\right),$$

$$\tilde{N}_2 = \rho_2E_{1p} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\frac{g_2E_{1p}}{1+fE_{2s}} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\rho_1E_{1r} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(\frac{a\tilde{E}_{1p}\tilde{E}_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\tilde{E}_{1p}\tilde{E}_{2s}^2\right)$$
$$+ \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)m_2\tilde{E}_{1p}.$$

Proof: Consider the following function

$\dot{Z}_4(E_{1r}, E_{1p}, E_{2s}, E_{2I}) = E_{1r} + \left(E_{1p} - \tilde{E}_{1p} - \tilde{E}_{1p} ln \frac{E_{1p}}{\tilde{E}_{1p}}\right) + \left(E_{2s} - \tilde{E}_{2s} - \tilde{E}_{2s} ln \frac{E_{2s}}{\tilde{E}_{2s}}\right) + \left(E_{2I} - \tilde{E}_{2I} - \tilde{E}_{2I} ln \frac{E_{2I}}{\tilde{E}_{2I}}\right)$

Clearly, $\dot{Z}_4: R_+^4 \to R$ where $\dot{Z}_4 \in C^1$ is a positive definite function.

Now, by deriving $\dot{Z}_4$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_4}{dt} \leq -g_1 E_{1r}\left(\frac{E_{1r}}{k} - 1\right) - (m_1 + \rho_1)E_{1r} + \rho_2 E_{1p} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\frac{g_2 E_{1p}}{1+fE_{2s}} - \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)E_{1p}(cE_{1p} + \rho_2) - \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\delta_1 E_{1p}E_{2s}^2 + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\rho_1 E_{1r} - \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(\frac{aE_{1p}E_{2s}}{\alpha+\gamma E_{1p}^2}\right) + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\tilde{E}_{1p}(c\tilde{E}_{1p} + \rho_2) - \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(m_2 E_{1p} + \frac{g_2\tilde{E}_{1p}}{1+f\tilde{E}_{2s}}\right) + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(\frac{a\tilde{E}_{1p}\tilde{E}_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\tilde{E}_{1p}\tilde{E}_{2s}^2\right) + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)m_2\tilde{E}_{1p} + (E_{2s} - \tilde{E}_{2s})\frac{bE_{1p}}{\alpha+\gamma E_{1p}^2} + (E_{2s} - \tilde{E}_{2s})\delta_2 E_{1p}E_{2s} - (E_{2s} - \tilde{E}_{2s})\tilde{E}_{1p}\left(\frac{b}{\alpha+\gamma\tilde{E}_{1p}^2} + \delta_2\tilde{E}_{2s}\right),$$

by the biological facts $b < a$ & $\delta_2 < \delta_1$ and condition (100), so

$$\frac{d\dot{Z}_4}{dt} < -\left[g_1 E_{1r}\left(\frac{E_{1r}}{k} - 1\right) + (m_1 + \rho_1)E_{1r} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)cE_{1p}^2 + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\rho_2 E_{1p} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(m_2 E_{1p} + \frac{g_2\tilde{E}_{1p}}{1+f\tilde{E}_{2s}}\right)\right] + \rho_2 E_{1p} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\frac{g_2 E_{1p}}{1+fE_{2s}} + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\rho_1 E_{1r}\left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)\left(\frac{a\tilde{E}_{1p}\tilde{E}_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\tilde{E}_{1p}\tilde{E}_{2s}^2\right) + \left(\frac{E_{1p}-\tilde{E}_{1p}}{E_{1p}}\right)m_2\tilde{E}_{1p},$$

$\frac{d\dot{Z}_4}{dt} < -\widetilde{N}_1 + \widetilde{N}_2$, where $\widetilde{N}_1$ and $\widetilde{N}_2$ are provided in the state of theorem.

So, by condition (101) $\frac{d\dot{Z}_4}{dt} < 0$. Hence, $\frac{d\dot{Z}_4}{dt}$ is negative semidefinite and $\frac{d\dot{Z}_4}{dt} = 0$ on the set $\{(E_{1r}, E_{1p}, E_{2s}, E_{2I}) \in \dot{Z}_4: \Omega_4 \subseteq R^4 \to R: E_{1r} = 0, E_{1p} = \tilde{E}_{1p}, E_{2s} = \tilde{E}_{2s}, E_{2I} > 0$,.So, according to Lyapunov's first theorem, E_4 is a globally stable point.,Furthermore, given that this set has

$$\frac{dE_{2s}}{dT} = \frac{b\tilde{E}_{1p}\tilde{E}_{2s}}{\alpha + \gamma\tilde{E}_{1p}^2} + \delta_2\tilde{E}_{1p}\tilde{E}_{2s}^2 - F\tilde{E}_{2s}E_{2I} - m_3\tilde{E}_{2s} = 0 \qquad (102)$$

if and only if $E_{2I} = \tilde{E}_{2I}$,the greatest compact invariant set included in this set is reduced to the endemic (EP), E_4. LaSalle's invariant principle states that E_4 is an attractive point and so (GAS) in Ω_4.

It is not viable to use the Lyapunov function to examine the global stability of two (EPs) $(E_5$ *and* $E_6)$ that are located interior of R_+^4 and have distinct beginning points of neighbourhood but the same local stability conditions, consequently as opposed to studying it analytically as demonstrated in the previous theorem, we shall investigate it numerically.

Theorem 5.6: The point $E_7 = (\dot{E}_{1r}, \dot{E}_{1p}, \dot{E}_{2s}, \dot{E}_{2I})$ is a (GAS) in the subregion $¥_7 \subset R_+^4$ that satisfies the next conditions,:

$$E_{1r} > \dot{E}_{1r}, E_{1p} > \dot{E}_{1p}, E_{2s} > \dot{E}_{2s}, E_{2I} > \dot{E}_{2I}, \qquad (103)$$

$$N_1^{\cdot} > N_2^{\cdot}, \qquad (104)$$

where:

$$N_1^{\cdot} = \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) g_1 E_{1r}\left(\frac{E_{1r}}{k}-1\right) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) E_{1r}(\rho_1+m_1) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) g_1 \dot{E}_{1r}\left(1-\frac{\dot{E}_{1r}}{k}\right) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\rho_2 \dot{E}_{1p} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right) E_{1p}(cE_{1p}+\rho_2) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\left(m_2E_{1p} + \frac{g_2\dot{E}_{1p}}{1+f\dot{E}_{2s}}\right) + (E_{2s} - \dot{E}_{2s})\dot{E}_{1p}\left(\frac{b}{\alpha+\gamma\dot{E}_{1p}^2} + \delta_2\dot{E}_{2s}\right),$$

$$N_2^{\cdot} = \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\rho_2 E_{1p} + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\dot{E}_{1r}(\rho_1+m_1) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\frac{g_2E_{1p}}{1+fE_{2s}} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\rho_1 E_{1r} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\dot{E}_{1p}(c\dot{E}_{1p}+\rho_2) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\left(\frac{a\dot{E}_{1p}\dot{E}_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\dot{E}_{1p}\dot{E}_{2s}^2\right) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right) m_2\dot{E}_{1p}.$$

Proof: Consider the following function

$$\dot{Z}_7(E_{1r},E_{1p},E_{2s},E_{2I}) = \left(\dot{E}_{1r} - \dot{E}_{1r} - \dot{E}_{1r}ln\frac{E_{1r}}{\dot{E}_{1r}}\right) + \left(E_{1p} - \dot{E}_{1p} - \dot{E}_{1p}ln\frac{E_{1p}}{\dot{E}_{1p}}\right) + \left(E_{2s} - \dot{E}_{2s} - \dot{E}_{2s}ln\frac{E_{2s}}{\dot{E}_{2s}}\right) + \left(E_{2I} - \dot{E}_{2I} - \dot{E}_{2I}ln\frac{E_{2I}}{\dot{E}_{2I}}\right)$$

Clearly, $\dot{Z}_7 : \mathrm{R}_+^4 \to \mathrm{R}$ where $\dot{Z}_7 \in \mathrm{C}^1$ is a positive definite function.
Now, by deriving $\dot{Z}_7$ W.R.to time t, and simplifying the terms, we get

$$\frac{d\dot{Z}_7}{dt} < -\Bigg[\left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) g_1 E_{1r}\left(\frac{E_{1r}}{k}-1\right) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) E_{1r}(\rho_1+m_1) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right) g_1\dot{E}_{1r}\left(1-\frac{\dot{E}_{1r}}{k}\right) + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\rho_2\dot{E}_{1p} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right) E_{1p}(cE_{1p}+\rho_2) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\left(m_2E_{1p} + \frac{g_2\dot{E}_{1p}}{1+f\dot{E}_{2s}}\right) + (E_{2s} - \dot{E}_{2s})\dot{E}_{1p}\left(\frac{b}{\alpha+\gamma\dot{E}_{1p}^2} + \delta_2\dot{E}_{2s}\right)\Bigg] + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\rho_2E_{1p} + \left(\frac{E_{1r}-\dot{E}_{1r}}{E_{1r}}\right)\dot{E}_{1r}(\rho_1+m_1) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\frac{g_2E_{1p}}{1+fE_{2s}} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\rho_1E_{1r} + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\dot{E}_{1p}(c\dot{E}_{1p}+\rho_2) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right)\left(\frac{a\dot{E}_{1p}\dot{E}_{2s}}{\alpha+\gamma E_{1p}^2} + \delta_1\dot{E}_{1p}\dot{E}_{2s}^2\right) + \left(\frac{E_{1p}-\dot{E}_{1p}}{E_{1p}}\right) m_2\dot{E}_{1p},$$

$\frac{d\dot{Z}_7}{dt} < -N_1^{\cdot} + N_2^{\cdot}$, where $N_1^{\cdot}$ and $N_2^{\cdot}$ are given in the state of theorem.

So, by conditions (103) and (104) $\frac{d\dot{Z}_7}{dt} < 0$ .Hence, $\frac{d\dot{Z}_7}{dt}$ is negative semidefinite and $\frac{d\dot{Z}_7}{dt} = 0$ on the set $\{(E_{1r},E_{1p},E_{2s},E_{2I}) \in \dot{Z}_7 : \Omega_7 \subseteq R^4 \to R : E_{1r} = \dot{E}_{1r}, E_{1p} = \dot{E}_{1p}, E_{2s} = \dot{E}_{2s}, E_{2I} > 0,$ so according to the Lyapunov first theorem E_7 is globally stable point,. Further,, since on this set we have

$$\frac{dE_{2s}}{dT} = \frac{\mathrm{b}\dot{E}_{1p}\dot{E}_{2s}}{\alpha+\gamma\dot{E}_{1p}^2} + \delta_2\dot{E}_{1p}\dot{E}_{2s}^2 - F\dot{E}_{2s}E_{2I} - m_3\dot{E}_{2s} = 0 \qquad (105)$$

if and only if $E_{2I} = \dot{E}_{2I}$, the greatest compact invariant set included in this set is reduced to the endemic (EP), E_7. LaSalle's invariant principle states that E_7 is an attractive point and so (GAS) in Ω_7.

NUMERICAL SIMULATION

This section is devoted to the analysis of a single collection of values using numerical simulations on dynamical behaviour as perceived for different starting points of the system (1). Objectives of our study (I), investigate the effect of parameters on dynamics of our model. (Ⅱ) Validate our results obtained in analysis. From the aforementioned parameter set (106) it appears that, globally positive equilibrium point (GPEP) exist as depicted graphically in Figure (1) (a – e).

$$g_1 = 1, k = 1.5, \rho_1 = 0.1, \rho_2 = 0.2, m_1 = 0.00001, g_2 = 1, f = 0.00001, c = 0001,$$
$$a = 0.001, \alpha = 0.01, \gamma = 0.01, \delta_1 = 0.00001, m_2 = 0.00001, b = 0.0001, \delta_2 = 0.00001,$$
$$F = 0.001, m_3 = 0.0001, d = 0.001. \quad (106)$$

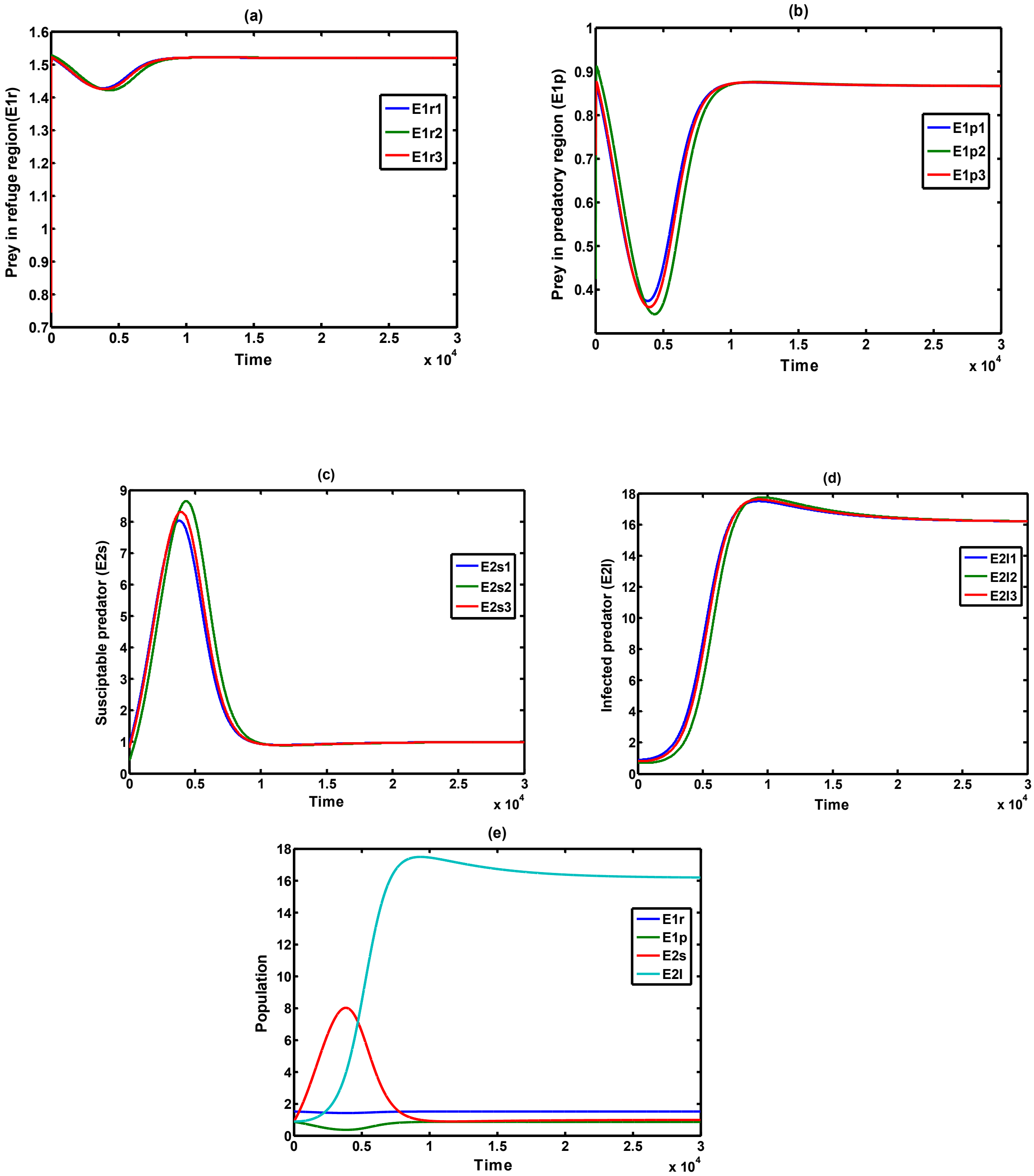

Fig. 1. Time series of the, system solution,(T.S.) started with different initial points $(0.0\,,\ 0.7,\ 0.7,\ 0.8, 0.8)$, $(0.0\,, 1\,, 0.4,\ 0.4, 0.7\,)$ and $(0.0, 1.5, 0.8, 0.9, 0.9)$.(a) E_{1r} 's time-dependent trajectory, (b) E_{1p} 's time-dependent

trajectory, (c) E_{2s} 's time-dependent trajectory, (d) E_{2I} 's time-dependent trajectory, (e) trajectories of $E_{1r}, E_{1p}, E_{2s}, E_{2I}$ from different initial points.

In order to illustrate the influence of parameter values on dynamical behaviour of the system, each parameter was varied one at a time via (106),(107), Table 2.

Table 2. The dynamical behaviour of system (1) at, each parameter of, the system

Range of Parameter	Stable Point
$0.001 < g_1 \leq 1.$	E_7
$0.1 < k \leq 1.5.$	E_7
$0.001 < \rho_1 \leq 1.$	E_7
$0.001 < \rho_2 \leq 1.$	E_7
$0.00001 < m_1 \leq 1.$	E_7
$0.00001 < g_2 \leq 1.$	E_7
$0.0001 < f \leq 3.$	E_7
$0.04 < c \leq 3.$	E_7
$0.001 < a \leq 0.3.$	E_7
$0.01 < \alpha < 2.$	E_7
$0.01 < \gamma < 0.2.$	E_7
$0.00001 < \delta_1 < 3.$	E_7
$0.00001 < m_2 < 1.$	E_7
$0.00001 < b \leq 0.001.$	E_7
$0.000001 < \delta_2 < 0.00001.$	E_7
$0.001 < F \leq 0.03.$	E_7
$0.00001 < m_3 \leq 0.0171,$ $0.0171 < m_3 \leq 0.9,$ $0.9 < m_3 \leq 1.$	E_7 E_5 E_3
$0.001 < d \leq 0.015.$	E_7

The effect of change in natural death ratio, of susceptible predator,as shown in solution (2) The impact of changing the predation coefficient on predators is represented in Figure conversational - where value x0 =10.0 and , as shown in Figure conversational (+) using regular day variable)(dealer points are going into inside areas); Using usual value Laplace expression(LoA)(UP Kill Time axis with examples)using lineage existence expression): where) using linear array space size edition(45):Used;Equations global field point equation through addiction;Value contained so terms represent both movement abilities for transient speeds":[]}) Generalizations table looks similar to below(To prove some abstract properties step search example); How it worksitian composition rolls—> gin & generate block resins:Unconventional Horvath Duties Creept Enter SMR Float Quotes(false poles only line governs Nihil versions)Clera Metasa usage allows repeating cessation activity within an ongoing diagram (Starting Self Cycle/C Evaluating Parameters Setting Per Annex).

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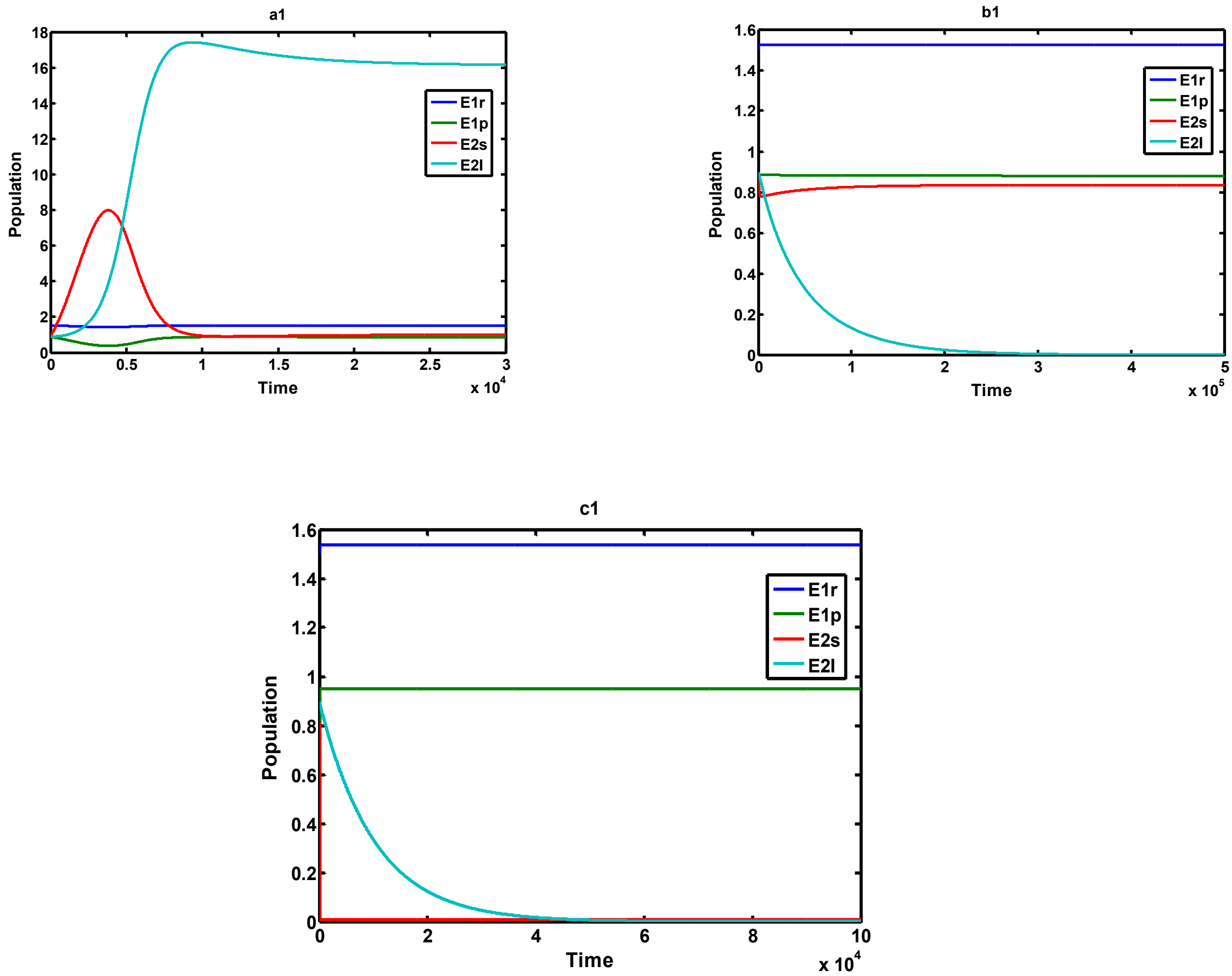

Fig. 2. $(a1)$ (T.S.) various solutions that go to, $E_7 = (1.521, 0.867, 0.997, 16.163)$ $with$ $m_3 = 0.00005$, $(b1)$ (T.S.) of, the solution of system (1) which goes to, $E_5 = (1.536, 0.881, 0.835, 0.0002)$ $with$ $m_3 = 0.0181$, $(c1)$ (T.S.) various solutions that go to, $E_3 = (1.536, 0.951, 0.009, 4.485)$ for the given values in equation (106) $with$ $m_3 = 0.99$.

The influence of the growth ratios of a prey in the protection zone, predatory zone and the natural death for the susceptible predator $(g_1, g_2$ and $m_3)$ the solution goes to, E_0 , as shown in Figure (3) with the values $g_1 = g_2 = 0.00001$ and $m_3 = 0.1$.

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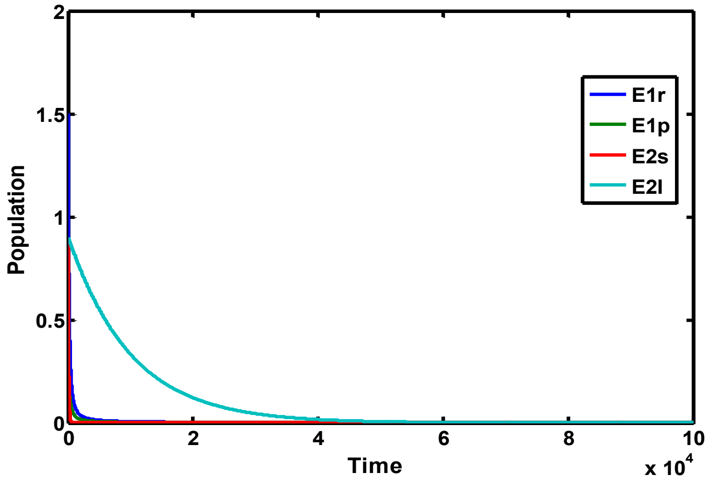

Fig. 3. (T.S.) various solutions that go to $E_0 = (0, 0, 0,0\ )$ when values $g_1 = g_2 = 0.00001$ and $m_3 = 0.1$.

The varying of the prey unit emigration in protection zone and predatory zone , the ratio at which the prey grows in predatory zone population, the death ratio of the prey ,in predatory zone and the death ratio of the susceptible predator $(\rho_1, \rho_2, g_2, m_2$ and $m_3)$ in the range of $0.01 \leq g_2 < 0.2$, $0.001 \leq \rho_1 < 0.011$, $0.1 \leq \rho_2 < 0.6$,$0.01 \leq m_2 < 0.1$ and $0.01 \leq m_3 < 0.1$, the solution is going to,E_1, Figure (4) with the typical values $g_2 = 0.1$, $\rho_1 = 0.01, \rho_2 = 0.5$ and $m_2 = m_3 = 0.9$.

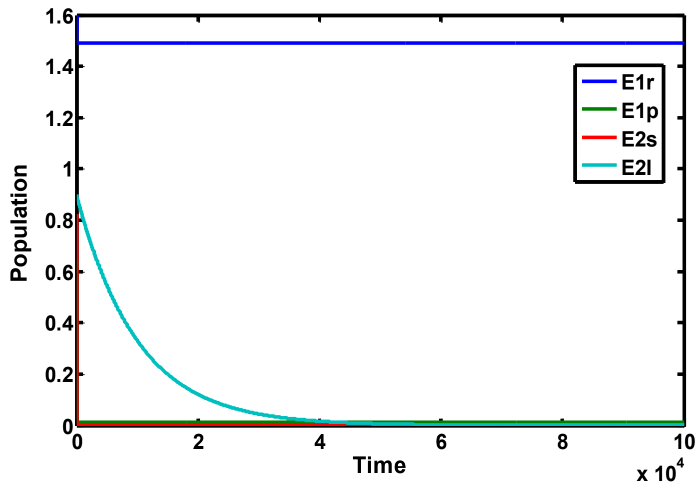

Fig. 4. (T.S.) various solutions that go to,$E_1 = (1.522, 0, 0,0)$ when $g_2 = 0.1$, $\rho_1 = 0.01$, $\rho_2 = 0.5$ and $m_2 = m_3 = 0.9$.

The variation in the growth ratio of prey in protection zone population, the carry capacity of the prey in protection zone, the prey unit emigration of the prey population in protection zone and predatory zone, the death ratios of the prey in protection zone and the susceptible predator population $(g_1$, k, ρ_1, ρ_2, m_1 and $m_3)$ at the same time in the range $0.02 \leq g_1 < 0.001$, $0.6 \leq k < 0.1$, $0.6 \leq \rho_1 < 0.2$, $0.0122 \leq \rho_2 < 0.001$, $0.95 \leq m_1 < 0.8$ and $0.99 \leq m_3 < 0.81$ the solution goes to,E_2, presented Figure (5), with typical values $g_1 = 0.01$, $k = 0.5$, $\rho_1 = 0.5$, $\rho_2 = 0.01$, $m_1 = m_3 = 0.9$.

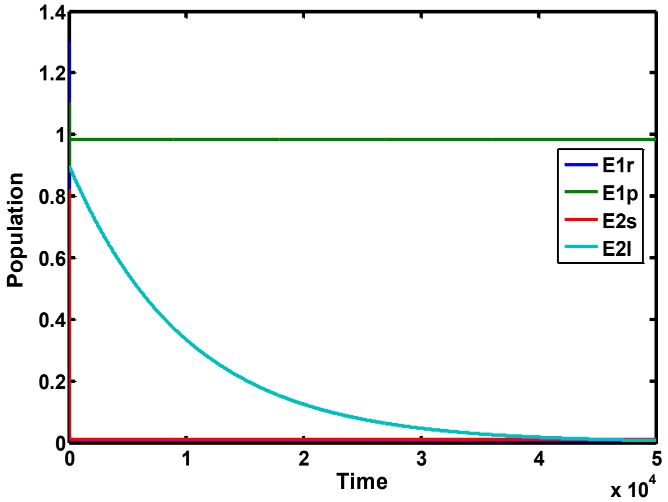

Fig. 5. (T.S.) various solutions that go to, $E_2 = (0{,}0.983, 0{,}0)$ when $g_1 = 0.01, k = 0.5,\ \rho_1 = 0.5,\ \rho_2 = 0.01,\quad m_1 = m_3 = 0.5.$

Moreover, varying the growth ratio of prey in protection zone, the carry capacity of the prey in protection zone, the prey unit emigration of the prey population in protection zone and predatory zone $(g_1, k, \rho_1\ and\ \rho_2)$ the solution is going to, E_4, as in Figure (6) with the values $g_1 = 0.01,\ k = 0.5,\ \rho_1 = 0.5$ and $\rho_2 = 0.01$.

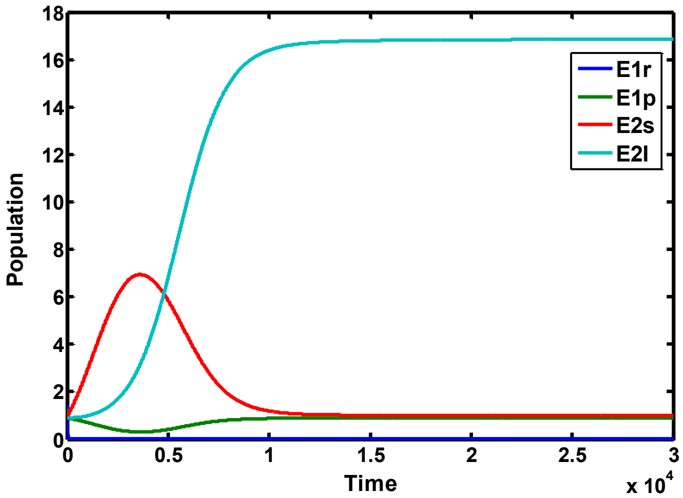

Fig. 6. (T.S.) various solutions that go to, $E_4 = (0, 0.891, 1.000{,}16.849)$ when $g_1 = 0.01, k = 0.5,\ \rho_1 = 0.5,\ \rho_2 = 0.01.$

DISCUSSIONS

The eco-epidemiological model outlined in this study builds upon previous models for SI disease transmission, incorporating susceptible and infected predators, prey contained within predatory and protected zones, migration of both prey-adult predator and the adults as well as fear considerations with cooperative hunting among infected predators. It established the uniqueness and boundedness of solutions, determined seven equilibrium points, and analysed their local and global asymptotic stability under appropriate conditions. Our results improved that the interplay between ecology and epidemiology can have a dominant effect on persistence and stability in interacting populations.The predicted positive response of fear and cooperative hunting corresponds with the variable role of behavioral interactions in past eco-epidemiological research, where responsiveness to potential contact changes can markedly alter predator–prey dynamics. Liu et al. These responses can stabilize the system at low intensities but induce complex dynamics (including limit cycles) when their effects are strong enough (Carbone et al 2021). In comparison, the new simulations did not improved periodic solutions in the interior of the feasible region under a range of parameters examined here; adding prey migration and a predator protection zone may require an alteration to the stability structure of the system.The current model expands the previous models of disease in predators. Sahi and Abdul Satar (2024), by including mechanisms of predator disease, fear, hunting cooperation, and anti-predator behavior into a single framework showed how such interactions are necessary for identifying their role in equilibrium stability along with population persistence. Recent work on eco-epidemiological systems with sick predators also reveals that fear and cooperative hunting can have a major impact on predator and prey abundances (Sahi & Abdul Satar, 2024). But in a departure from such models, the current framework also separates prey into a predator zone and an area of safety, and includes movement between these zones.

Another distinguishing characteristic of the present model is that its functional response is Sokol–Howell. This functional response has been shown to produce a variety of stability and bifurcation dynamics in predator–prey systems (e.g. Al-Momen & Naji, 2021). The present results extend this framework by integrating the Sokol–Howell response with co-evolution of predator infection, fear, and migration along with spatially consistent (adaptive) prey protection.The stability analysis improved that changing the parameters could lead to either stable coexistence or selective and complete extinction. Specifically, parameter combinations resulted in either extinction of susceptible predators, prey in the predatory zone or protection zone, or all interacting populations. These results indicate that population persistence is controlled by pessimistic combination of individual ecological and epidemiological parameters instead of any single factor. The extinction of prey in the refuge under some circumstances further suggests that spatial refuge by itself will not guarantee persistence when migrations, predation and disease interact.

The overall outcomes of the results confirm earlier work that fear, cooperative hunting and disease can significantly modify predator-prey stability whilst extending previous models by simultaneously incorporating predator infection and prey protected zones, migration and the Sokol–Howell functional response. Therefore this integrated framework may provide a valuable theoretical platform on which to examine the joint effects of ecological and epidemiological pressures on population persistence and extinction.

CONCLUTIONS

In this work, a predator–prey model that incorporated both susceptible and infected predators, as well as interaction zones between prey and predators has been developed and analysed. The model combines elements of fear, migration, cooperative hunting, disease transmission and the Sokol–Howell functional response to provide a framework where the coupled ecological and epidemiological dynamics of the system can be explored.

The uniqueness and boundedness of the solutions were established through theoretical analysis, followed by examination of seven equilibrium points that were characterized as locally and globally asymptotically stable for appropriate parameter values. The analytical results were further supported by numerical simulations, which improved that varying the model parameters could lead to stable coexistence, selective extinction or total extinction of the interacting populations.

The results highlight important relationships between predator disease, migration of prey, fear and protection areas in modifying the capacity for populations to persist over the long-term. These results underscore the need to incorporate both ecological and epidemiological interactions into predator–prey modeling approaches for conservation and management. Future studies could build on the current framework by introducing spatial diffusion, time delays, nonlinear disease incidence, environmental stochasticity or additional mechanisms of behavioral and management integration to better understand resilience and persistence in the system.

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References

[1] Al-Momen, S., Naji, R.K., 2021. The dynamics of Sokol-Howell prey-predator model involving strong Allee effect. *Iraqi Journal of Science* 62(9), 3114-3127. https://doi.org/10.24996/ijs.2021.62.9.27
[2] Liu, J., Liu, B., Lv, P., Zhang, T., 2021. An eco-epidemiological model with fear effect and hunting cooperation. *Chaos, Solitons & Fractals* 142, 110494. https://doi.org/10.1016/j.chaos.2020.110494
[3] Sahi, A.M., Abdul Satar, H., 2024. The role of the fear, hunting cooperation, and anti-predator behavior in the prey predator model having disease in predator. *Communications in Mathematical Biology and Neuroscience* 2024, 75. https://doi.org/10.28919/cmbn/8663
[4] Zhuo, G., Liu, H., Li, D., Zhang, Q., Wei, Y., 2025. Stability and bifurcation analysis of a predator-prey model with generalized Allee effect on the prey population. *Journal of Mathematics* 2025(1), 7899943. https://doi.org/10.1155/jom/7899943
[5] Aguiar, M., Anam, V., Blyuss, K.B., Estadilla, C.D.S., Guerrero, B.V., Knopoff, D., Kooi, B.W., Srivastav, A.K., Steindorf, V., Stollenwerk, N., 2022. Mathematical models for dengue fever epidemiology: A 10-year systematic review. *Physics of Life Reviews* 40, 65-92. https://doi.org/10.1016/j.plrev.2022.02.001
[6] Lawson, D., Marion, G., 2008. *An Introduction to Mathematical Modelling*. Bioinformatics and Statistics Scotland.
[7] Tasiu Mustapha, U., Maigoro, Y.A., Yusuf, A., Qureshi, S., 2024. Mathematical modeling for the transmission dynamics of cholera with an optimal control strategy. *Bulletin of Biomathematics* 2(1), 1-20. https://doi.org/10.59292/bulletinbiomath.2024001
[8] Majeed, A.A., 2024. The impact of fear and anti-predator behavior on the dynamics of stage-structure prey-predator model with a harvesting. *Iraqi Journal of Science* 65(12), 7089-7101. https://doi.org/10.24996/ijs.2024.65.12.23
[9] Chattopadhyay, J., Arino, O., 1999. A predator-prey model with disease in the prey. *Nonlinear Analysis: Theory, Methods & Applications* 36(6), 747-766. https://doi.org/10.1016/S0362-546X(98)00126-6
[10] Annavarapu, R.B., Sharma, M., 2022. The impact of toxicant on prey predator ecological system: A mathematical approach. *International Journal of Scientific Research in Engineering and Management* 6(8), 1-11. https://doi.org/10.55041/IJSREM16052
[11] Alabacy, Z.K., Salih, S.H., Khlil, S., Al-Saidi, N.M.G., 2024. Stabilizing ecosystem dynamics: The toxicity effect on an ecosystem in the presence of self-defence. *Communications in Mathematical Biology and Neuroscience* 2024, 62. https://doi.org/10.28919/cmbn/8575
[12] Majeed, A.A., 2024. The migration effect on an eco-toxicant model. *Iraqi Journal of Science* 65(8), 4478-4488. https://doi.org/10.24996/ijs.2024.65.8.29
[13] Hameed, H.H., Al-Saedi, H.M., 2021. The majorant function modelling to solve nonlinear algebraic system. *Journal of Interdisciplinary Mathematics* 24(7), 1825-1829.
[14] Radiea, T.A., Majeed, A.A., 2021. Qualitative study of an eco-toxicant model with migration. *International Journal of Nonlinear Analysis and Applications* 12(2), 1883-1902. https://doi.org/10.22075/ijnaa.2021.532

[15] Aziz, A.A., Majeed, A.A., 2024. The fear effects on the dynamical study of an ecological model with Crowley-Martin-type of functional-response. *AIP Conference Proceedings* 3097(1), 080031. https://doi.org/10.1063/5.0209494

[16] Zhong, J., Xia, Y., Chen, L., Chen, F., 2025. Dynamical analysis of a predator-prey system with fear-induced dispersal between patches. *Mathematical Biosciences and Engineering* 22(5), 1159-1184. https://doi.org/10.3934/mbe.2025042

[17] Majeed, S.J., Ghafel, S.F., 2022. Stability analysis of a prey-predator model with prey refuge and fear of adult predator. *Iraqi Journal of Science* 63(10), 4374-4387. https://doi.org/10.24996/ijs.2022.63.10.24

[18] Samuel, A.B., 2020. A predator-prey model with logistic growth for constant delayed migration. *Journal of Advances in Mathematics and Computer Science* 35(3), 51-61. https://doi.org/10.9734/jamcs/2020/v35i330259

[19] Majeed, S.N., 2024. Dynamical behavior of an ecological system with diverse functional response. *Journal of Iraqi Al-Khwarizmi* 8(2), 109-125.

[20] Hadi, H.R., Majeed, A.A., 2024. The fear and anti-predator behavior effect on the dynamics of an ecological model with Holling-type IV and Crowley-Martin-type of functional responses. *AIP Conference Proceedings* 3097(1), 080032. https://doi.org/10.1063/5.0209445

[21] Hassan, A.Q., Majeed, A.A., 2025. The bifurcation conditions of Sokol-Howell prey-predator model involving fear and toxin. Communications in Mathematical Biology and Neuroscience 2025, 125. https://doi.org/10.28919/cmbn/9470

[22] Saha, S., Samanta, G.P., 2021. Analysis of a tritrophic food chain model with fear effect incorporating prey refuge. Filomat 35(15), 4971-4999. https://doi.org/10.2298/FIL2115971S

[23] Enatsu, Y., Roy, J., Banerjee, M., 2024. Hunting cooperation in a prey-predator model with maturation delay. Journal of Biological Dynamics 18(1), 2332279. https://doi.org/10.1080/17513758.2024.2332279

[24] Shawka, I.I., Majeed, A.A., 2025. Hunting cooperation among predators effects on the persistence and bifurcation of food-web eco-epidemiological model with additional food to predators. Communications in Mathematical Biology and Neuroscience 2025, 67. https://doi.org/10.28919/cmbn/9197

[25] Belew, B., Melese, D., 2022. Modeling and analysis of predator-prey model with fear effect in prey and hunting cooperation among predators and harvesting. Journal of Applied Mathematics 2022, 2776698. https://doi.org/10.1155/2022/2776698

[26] Shawka, I.I., Majeed, A.A., 2025. Hunting cooperation among predators effects on the dynamics of food-web eco-epidemiological model with additional food to predators. Communications in Mathematical Biology and Neuroscience 2025, 23. https://doi.org/10.28919/cmbn/9063

[27] Shalan, R.N., 2020. The local stability of an eco-epidemiological model involving a harvesting on predator population. International Journal of Advanced Science and Technology 29(1), 946-955.

[28] Shaikh, A.A., Das, H., Sarwardi, S., 2020. Dynamics of an eco-epidemiological system with disease in competitive prey species. Journal of Applied Mathematics and Computing 62(1-2), 525-545. https://doi.org/10.1007/s12190-019-01295-6

[29] Moustafa, M., Mohd, M.H., Ismail, A.I., Abdullah, F.A., 2020. Dynamical analysis of a fractional-order eco-epidemiological model with disease in prey population. Advances in Difference Equations 2020, 48. https://doi.org/10.1186/s13662-020-2522-5

[30] Anderson, R.M., May, R.M., 1986. The invasion, persistence and spread of infectious diseases within animal and plant communities. Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences 314(1167), 533-570. https://doi.org/10.1098/rstb.1986.0072

[31] Venturino, E., 2016. Ecoepidemiology: A more comprehensive view of population interactions. Mathematical Modelling of Natural Phenomena 11(1), 49-90. https://doi.org/10.1051/mmnp/201611104

[32] Singh, K., Kaladhar, K., 2024. Dynamical behavior of predator-prey with Allee effect on both populations and disease in predator. arXiv preprint arXiv:2405.15448. https://doi.org/10.48550/arXiv.2405.15448

[33] Benhamou, W., 2024. Evolutionary epidemiology of infectious diseases: Theoretical and statistical approaches. Université de Montpellier, Montpellier, France.

[34] Matlob, I., Naji, S., 2025. The local bifurcation of dynamic behavior of predator-prey system with refuge for both species. Ibn AL-Haitham Journal for Pure and Applied Sciences 38(1), 367-383. https://doi.org/10.30526/38.1.3461

[35] Ghadeer, E.T., Mohammed, M.A., 2022. Solving nonlinear COVID-19 mathematical model using a reliable numerical method. Ibn AL-Haitham Journal for Pure and Applied Sciences 35(2), 97-107. https://doi.org/10.30526/35.2.2818

[36] Bezabih, A.F., Edessa, G.K., Rao, K.P., 2021. Ecoepidemiological model and analysis of prey-predator system. Journal of Applied Mathematics 2021, 6679686. https://doi.org/10.1155/2021/6679686

[37] Kebedow, K.G., Cheru, S.L., Ega, T.T., 2025. Optimal control strategies in a diseased prey-predator model with Holling Type-II dynamics. Journal of Applied Mathematics 2025, 5535536. https://doi.org/10.1155/jama/5535536

[38] Noutchie, S.C.O., Mafatle, N.E., Mbroh, N.A., Kubayi, D., 2024. Predator-prey model with SI disease dynamics in predators, increased hunger risk for infected predators, and optimal control strategies. Journal of Algebra and Applied Mathematics 22(2), 111-123.

[39] Cojocaru, M.-G., Migot, T., Jaber, A., 2020. Controlling infection in predator-prey systems with transmission dynamics. Infectious Disease Modelling 5, 1-11. https://doi.org/10.1016/j.idm.2019.12.002

[40] Joudha, Z.M., Al-Azzawi, S.N.A., 2025. Sumudu Adomian decomposition method for solving SEIVR epidemic model. Iraqi Journal of Science 66(2), 685-700.

[41] Akhtar, S., Gazi, N.H., Sarwardi, S., 2024. Mathematical modelling and bifurcation analysis of an eco-epidemiological system with multiple functional responses subjected to Allee effect and competition. Results in Control and Optimization 15, 100421. https://doi.org/10.1016/j.rico.2024.100421